PONTIFICIAE ACADEMIAE SCIENTIARVM SCRIPTA VARIA

28

SEMAINE D'ETUDE

SUR

LE ROLE DE L'ANALYSE ECONOMETRIQUE DANS LA FORMULATION DE PLANS DE DEVELOPPEMENT

7 - 13 octobre 1963

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STUDY WEEK

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A NEW THEORETICAL APPROACH TO THE PROBLEMS OF ECONOMIC GROWTH

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I consider myself very fortunate to be given the opportunity of presenting this work of mine to a group of so eminent economists. At the same time, I feel I am at some disadvantage because I am not inserting myself into the prevailing line of contemporary economic thought. I am rather trying a different approach altogether and I may be faced with some lack of communication.

It may be useful, therefore, before I come to what I might call a multi-sector model of economic growth, to make some efforts to sketch out a few introductory remarks on the general line of thought within which I propose to move.

Chapter I (*)

Introduction

1. *The historical background of economic analysis*

I shall begin with a few historical remarks which might sound at first rather general and far-fetched but which will soon turn out to have a justification.

If we consider the historical context in which economic analysis has come into being, we may say that this context is represented by the *modern* world; namely by the stage of our history which is known as the age of experiment and science, because of the dominating idea that man, by using his own critical intellect, by observing nature and experimenting, can learn in a systematic way and can pass on his improved knowledge to the following generations.

In economic terms, the direct consequence has been a process of unprecedented increase of material wealth. The process may be distinguished, for analytical purposes, into two relevant phases, which we may call the phase of trade and the phase of industry. There is no clear-cut distinction between the two, as they have a common origin and are intermingled, but they appear nevertheless with very definite characteristics on the historical scene.

(*) The present work is a summarized version of a Ph. D. dissertation submitted by the author at the University of Cambridge in September 1962. Chapters II to VI have appeared already, as a publication for limited circulation, under the title *A Multi-sector Model of Economic Growth*, King's College, Cambridge, July 1963. Acknoweledgements for criticism and Comments are gratefully due to: R.M. Goodwin, N. Kaldor, J. Robinson, R.F. Kahn, J.S. Duesenberry, D.G. Champernowne, I.M.D. Little. Responsibility, of course, is entirely mine.

The phase of trade is the first to break through. It can be perceived even as early as at the turn of the first millennium, but more clearly later on, after the Renaissance « opening of the minds » towards the outside world. A few important improvements in the technique of transportation lead to discoveries of new lands and extend the horizon of the known world to include countries with climates and products previously unknown. New possibilities of trade open up, with a striking impact on the economic conditions of the whole world. The trading nations are suddenly better off, not because of a rise in world production, but because of a better utilisation of the production which already takes place. Each nation keeps her own institutions and organisational structure of production, but now she can advantageously exchange the products which are proper to her particular climate or localized resources for products which she could never produce or which she could produce only at much higher costs. The material wealth of all peoples is increased just by exchange, by a better spatial allocation of existing resources and products. This is the merchant era, an era which represents perhaps the most outstanding example of how all people can gain from trade.

Much slower to manifest itself is the phase of industry, which requires already, and thus presupposes, trade. Industry is a process of augmenting wealth through a material increase in the quantity and number of products, to be reached by the practical application of the advances of science, division and specialisation of labour, better organisation, invention and utilization of new sources of energy and new materials. Unlike trade, industry requires changes in the organisational structure of society. Therefore, it comes about slowly; but progressively. In fact, it requires long and painful social changes in the relations between men and the means of production before it can fully break out in the English « industrial revolution » of the eighteenth century. Of course, trade remains the natural and necessary complement of industry but, as a cause of *further* increments of wealth, it is bound to subside. Industry, on the other hand, is bound to remain a permanent cause of increase of wealth and to become pre-eminent as time goes on, owing to the very nature of its cumulative process.

These two aspects of the modern world seem to me very helpful in indicating the directions in which the emergence of the modern era has stimulated economic analysis.

The concept of trade is, so to speak, a *static* concept. It is associated with a situation in which a plurality of economic systems (or of individuals) are endowed with particular resources or products and try to gain advantages by exchange. The interest that such a situation arouses in an economist concerns the problem of how to reach the best allocation of given resources, namely of how to make the best use of what one has already. We may imagine a stationary situation in which a plurality of economic systems have reached equilibrium internally, but do not trade among themselves; and then another stationary situation in which the same economic systems, besides having reached an internal equilibrium, also trade with one another. Is is easy to show that the passage from the first to the second situation — i.e. a once-for-all change from no trade to a new situation of trade, to be maintained thereafter — normally brings about a gain for all. The problem involved is a *problem of rationality,* which may be expressed by a mathematical function to be maximised under certain constraints.

The concept of, and the problems entailed by, industry are quite different. Industry is, so to speak, a *dynamic* concept. It means production, i.e. the engagement and the application of man's ingenuity to make and shape the products he wants. But since by doing and experiencing man learns, it is implied in the very nature of carrying on a production activity that new and better methods of production will be discovered. Of course, to find new methods takes time, and takes time in a persistent way. The economist is faced here no longer with a problem of rationality, but with a *process of learning*. Any mathematical formulation of it cannot but be in terms of functions of time, since the process makes short steps, and may appear quite negligible, in the short run; but as it goes on incessantly, it is inevitably bound to become the more pronounced the longer the period which is considered. The contrast with the simple concept of trade is now evident. The passage from a position of no trade to a position of trade means a jump, which may be quite big but which is temporary, as it ends when the new equilibrium situation has been reached. The process of learning associated with industry, on the other hand, means a persistent movement, not a once-for-all change, but a *rate of change* in time — a movement which is cumulative and indefinite. In this sense, industry comes to realise most properly the concept of *progress* which is inherent in modern society.

Clearly, these are two distinct series of problems. A particularly important difference between the two, for theoretical analysis, is that they acquire an opposite practical relevance in relation to time, the former being relevant (in the short run) just when the latter is practically irrelevant and the latter becoming relevant just when (in the long run) the former becomes irrelevant. Both series of problems have been of course considered, in the course of development of economic thought. But according as to whether the economic theorists have been mainly impressed by the former or by the latter, the attitude they have taken to the type of hypotheses to choose has been diametrically opposite.

2. *Scarcity versus learning in economic analysis*

There are clear indications — it seems to me — that the Classical economists, writing under the strong impact of the English « industrial revolution », were well aware of the two aspects of the modern world mentioned above and had in fact no hesitation in singling out the industrial aspect as by far the more important.

At the very beginning of his *Wealth of Nations,* ADAM SMITH takes great pain in pointing out that it is « the skill, dexterity and judgment with which labour is applied » which is by far the pre-eminent factor accounting for the wealth of nations, « whatever be the soil, climate or extent of territory » (1). DAVID RICARDO in his turn, again on the very first pages of his *Principles,* sets out this opposition in terms of types of commodities. « There are some commodities — he says — the value of which is determined by scarcity alone. » These are the goods which are given by nature. We may call them the commodities of the *scarcity type.* RICARDO says that they « form a very small part of the mass of commodities daily exchanged in the market. » And he continues: « By far the greatest part of those goods, which are the object of desire, are produced by labour; and they may be multiplied, not in one country alone, but in many, almost without any assignable limit, if we are disposed to bestow the labour necessary to obtain them. » (2) We may call these, the commodities of the *production type.* It is on these commodities that RICARDO concentrated his analysis.

The whole economic theory that followed preferred, on the other hand, to abandon this approach and to go on to taking exactly the opposite view. The stream of economic thought which came to dominate the second part of last century (Marginalism) and which still now-a-days provides the backbone of most of contemporary economics, appears as a tendency to concentrate on the other type of commodities: the commodities

(1) ADAM SMITH, *An Inquiry into the Nature and Causes of the Wealth of Nations,* ed. by E. Cannan, p. 1.

(2) DAVID RICARDO, *On the Principles of Political Economy and Taxation,* p. 12. The references are to the edition by PIERO SRAFFA, with the collaboration of M.H. DOBB, *The Works and Correspondence of David Ricardo,* in 10 vls., Cambridge, 1951.

of the scarcity type. As we know, the typical marginalistic scheme of general equilibrium is a model of what has been called a *pure exchange* economy. (3). The model presupposes the existence of given natural resources in fixed quantities, and of a given number of individuals (owning the resources) with well-defined utility preferences. The economic problem these individuals have is one of rational choice. They have to find those prices (equilibrium prices) which bring about, through exchange, an optimum allocation of the given resources relatively to their original ownership-distribution. The problem can be represented analytically by a mathematical function which is being maximized, subject to certain constraints fixed by nature.

Of course, marginalist economists have then gone on to investigating all other economic problems as well (production included). The relevant point is, however, that they have done so by an *extension* of the theory they had originally developed for scarce commodities. Professor SAMUELSON, at the very beginning of what is one of the most rigorous versions of marginal economic analysis, claims exactly this. He claims to have been able to isolate a simple theory which can be applied to every economic investigation: a mathematical function to be maximized under given constraints (4).

This has been a crucial step. For, in this way, every single corner of economic theory has come to be permeated by the character of scarce goods and by the rational problem of making the best use we can of them, at the expense of the character of produced goods and of the learning process of human beings. It has meant that modern economists have

(3) See, for example: J.L. MOSAK, *General Equilibrium Theory in International Trade*, Bloomington, Indiana, 1944. The same scheme can be found in the very first chapters of all standard treatises of marginal economics. (See, for example: V. PARETO, *Cours d'économie politique*. Lausanne, 1895; J.R. HICKS, *Value and Capital*, Oxford, 1939; P.A. SAMUELSON, *Foundations of Economic Analysis*, Cambridge Mass., 1947).

(4) P.A. SAMUELSON, *op. cit.*, pp. 1 and ff.

been led to look at *the whole* of economic reality through the lens of a scarcity scheme, and thus to magnify or shrink the various aspects of reality according as to whether they do or do not fit into the pattern of a world of scarce goods.

I have thought it may not be altogether useless — after a century of marginal economic analysis — to go back to explore the possibilities, which may have remained unexploited, inherent into the other approach to economic reality, which has been left into oblivion since the time of the Classics.

3. *A pure production model*

It is my purpose to develop, in the following pages, a theoretical model of economic growth for an industrial economic system. In the whole theory, a central rôle will be played by the learning process of human beings, in its twofold aspects of technical improvements and of consumers' preference evolution. Scarce resources will not be considered, although of course this does not mean that they do not exist. It only means that the theory will be developed independently of any rational problem concerning their best utilization. All commodities considered are produced, and can be made practically in whatever quantity may be wanted, provided that they are devoted to the amount of efforts they technically require. Limitations of course exist, but not in the material world: they only reside in the knowledge and power of activity of Men.

As will be realised, this means adopting a procedure which is exactly opposite, though symmetrical, to the one followed by marginal analysis. The scheme itself might be called a *pure production* model, as against the pure exchange model of marginal economics. It will refer to a certain type of commodities (this time the commodities of the production type), will centre around a definite problem (the problem of production), and will be dominated by a general principle (represented by the learning process of human beings) (5).

There is one important respect, however, on which the following analysis will not take a path symmetrical to the one followed by marginal analysis. The model will be kept at a sufficiently high level of generality, so as to remain neutral with respect to the institutional organization of society. The preoccupation is that of singling out the necessary conditions for efficiency and equilibrium growth — conditions which will emerge as independent of any particular institutional set-up that society might choose to adopt.

(5) It may be useful to add a warning to the reader at this point. To take human activity, i.e. labour, as the only *non-produced* factor of production must not be interpreted as meaning that « labour is the only *scarce* factor », as — I have noticed — some of my friends, while discussing the present work with me, have tended to do. Such an interpretation would be incorrect: scarcity presupposes some aim the attainment of which is limited by the existing quantity of the factor. But no such aim, limited by the existing quantity of labour, is present here. As the reader will see presently, the systems of equations which will be considered yield solutions for *relative* prices and *relative* quantities. To say, in such a context, that labour is scarce has no sense.

CHAPTER II

THE PROCESS OF PRODUCTION IN THE SHORT RUN

1. *A very simple case: production by means of labour alone*

We may begin our analysis by considering the process of production at a given point of time or, to be more realistic, within a *short period of time*, defined in such a way that, within it, changes in population, productive capacity, technical knowledge and consumers' preferences are negligible.

The economic system we are considering is *closed* and consists of a society of individuals whose purpose is to produce goods in order to derive enjoyment (or relief from pain) by consuming them. Two types of activities are therefore performed in the system: a production activity and a consumption activity. At the beginning of the period, the production processes are programmed in the best way that is technically known and, at the end of the production processes, the commodities are consumed. Technical knowledge is supposed to be quite advanced so that each process of production requires division of labour and a marked specialisation. Therefore, each individual consumes only a very small part of the commodities he produces (or contributes to produce); and obtains all the others through exchange. Total production (and therefore consumption) is well diversified: the system produces many types of commodities, according to the preferences of its individual members, or consumers.

In order to keep the analysis in as simple terms as possible only *final* commodities will be considered. No intermediate stage will be explicity represented. After all, it is always possible, when needed, to re-introduce intermediate stages and intermediate commodities by a simple linear transformation, as will be shown and discussed in detail later on, in chapter VI. For the time being, therefore, all production processes will be considered as vertically integrated, in the sense that all their inputs are exclusively represented by services from two types of factors of production: labour and capital.

To begin with, however, it will be useful, *as a purely expository device,* to take a very simple step and to develop first a theoretical model (sections 2 and 3) where production is exclusively carried out by labour. Capital, as a factor complementary to labour, will then be introduced from section 4 on.

2. *The flows of commodities and of labour services, in physical terms and at current prices.*

Even in a system as simple as the one now described, there is a whole series of flows — or rather two different series of flows — which take place inside the period considered: flows of labour services from the individuals as labourers to the production processes, and flows of commodities from the production processes to the individuals as consumers. Suppose that the number of final commodities produced is $(n - 1)$. Then, since we consider no intermediate stage, there is a production process, behind each final commodity, which goes right back to the original factors of production: labour in our case. We have, therefore, $(n - 1)$ production processes or *sectors,* each of which consists of one labour input and of one product output.

All the individuals may then be considered as grouped in a final sector, which may be called household — sector n — which receives all productions for consumption and provides all labour services for production.

All these flows may be framed in a usual input-output table, which in our case becomes very simple. As is well known, the table can be looked at from two different points of view, and accordingly represented by two systems of identities. From a *physical* point of view, the production of each commodity is identically equal to the sum of all quantities of that commodity which are delivered as inputs to the other sectors. (In our simple case this sum is reduced to one term: the amount delivered to the household sector). Moreover, the sum of all labour services is identically equal to labour employed. Similarly, from a value point of view, the production of each commodity must be equal in value to the sum of the values of its total inputs, and the sum of the values of all commodities must be equal to the total income which is distributed to the factors of production (labour in our case).

We have therefore:

$$
\text{(II.1)}\qquad \left\{
\begin{array}{lllllll}
X_1 & & & & -x_{1n} & = & 0 \\
& X_2 & & & -x_{2n} & = & 0 \\
& & & & \cdot & \cdot & \cdot \\
& & & & \cdot & \cdot & \cdot \\
& & & & \cdot & \cdot & \cdot \\
& & & & \cdot & \cdot & \cdot \\
& & & & -x_{n-1,n} & = & 0 \\
-x_{n1} & -x_{n2} & \cdot\ \cdot\ \cdot\ \cdot & -x_{n,n-1} & +X_n & = & 0,
\end{array}
\right.
$$

and

$$
\text{(II.2)} \quad \begin{cases}
X_1 P_1 & & & -x_{n1} P_n = 0 \\
& X_2 P_2 & & -x_{n2} P_n = 0 \\
& & & \cdot \quad \cdot \quad \cdot \\
& & & \cdot \quad \cdot \quad \cdot \\
& & & \cdot \quad \cdot \quad \cdot \\
& & & \cdot \quad \cdot \quad \cdot \\
-x_{1n} P_1 - x_{2n} P_2 & \ldots & \ldots & +X_n P_n = 0 ,
\end{cases}
$$

where:

X_j = physical quantities produced, $j = 1, 2, \ldots (n-1)$;
P_i = prices, $i = 1, 2, \ldots n$;
X_n = quantity of labour used in all productive activities;
x_{ij} = physical flow from sector i to sector j.

It appears now very useful, for reasons of symmetry, to put $a_{ij} = \frac{x_{ij}}{X_j}$. Then, systems (II.1)-(II.2) may be expressed in matrix notation [1]:

$$
\text{(II.3)} \quad \begin{bmatrix}
-1 & 0 & \ldots & 0 & a_{1n} \\
0 & -1 & 0 \ldots & 0 & a_{2n} \\
 & & & & \cdot \\
 & & & & \cdot \\
0 & 0 & \ldots & -1 & a_{n-1,n} \\
a_{n1} & a_{n2} & \ldots & a_{n,n-1} & -1
\end{bmatrix}
\begin{bmatrix}
X_1 \\ X_2 \\ \cdot \\ \cdot \\ X_{n-1} \\ X_n
\end{bmatrix}
=
\begin{bmatrix}
0 \\ 0 \\ \cdot \\ \cdot \\ 0 \\ 0
\end{bmatrix}
$$

---

(1) Here and in all subsequent analysis, I shall be using matrices to represent the systems of linear equations (with the advantage of showing immediately the symmetrical properties). However, any compact notation will be avoided and matrices will always be written in full. In this way, the following analysis will require from the reader only a few elementary notions of matrix algebra.

$$
\text{(II.4)} \quad \begin{bmatrix} -1 & 0 & . & . & . & 0 & a_{n1} \\ 0 & -1 & 0 & . & . & 0 & a_{n2} \\ & & & & & & . \\ & & & & & & . \\ & & & & & & . \\ & & & & & & . \\ 0 & 0 & . & . & . & -1 & a_{n,n-1} \\ a_{1n} & a_{2n} & . & . & & a_{n-1,n} & -1 \end{bmatrix} \begin{bmatrix} P_1 \\ P_2 \\ . \\ . \\ . \\ . \\ P_{n-1} \\ P_n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ . \\ . \\ . \\ . \\ 0 \\ 0 \end{bmatrix}
$$

The $(a_{n1}, a_{n2}, \ldots . a_{n,\,n-1})$ represent technical coefficients of production and the $(a_{1n}, a_{2n}, \ldots . a_{n-1,\,n})$ represent demand coefficients of consumption. An evident property of both these systems is that they have the same, but transposed, matrix of coefficients. As they have been defined, they are nothing but an algebraic representation of the flows which take place in the economic system under examination. However, they can be looked at in a different way. If we consider the a_{ij}'s as parameters, then (II.3) and (II.4) form two systems of equations, and we can enquire into the nature of their solutions. These two systems are of a particular kind — they are linear and homogenous. Therefore, in order that they may have non-trivial solutions (i.e. solutions which are not all equal to zero), the coefficient matrix must be singular, namely it must satisfy the following condition (which is the same for both systems) (²):

$$
\begin{vmatrix} -1 & & & & & a_{1n} \\ & -1 & & & & a_{2n} \\ & & & & & . \\ & & & & & . \\ & & & & & . \\ & & & & -1 & a_{n-1,n} \\ a_{n1} & a_{n2} & . & . \; . \; . \; . & a_{n,n-1} & -1 \end{vmatrix} = 0.
$$

(²) For shortness, the zeros will not be explicitly written from now on; so that all entries of our matrices which are left blank should be interpreted as zeros.

By developing this determinant, the condition may be stated more simply as:

$$\text{(II.5)} \qquad a_{n1}\, a_{1n} + a_{n2}\, a_{2n} + \dots + a_{n,n-1}\, a_{n-1,n} - 1 = 0 \ .$$

If this condition is satisfied, then each of the two linear homogenous systems gives solutions for $(n-1)$ variables, while the n^{th} variable can be fixed arbitrarily. In our case, there evidently is for system (II.3) a quantity which is fixed, namely $\overline{X}_n$: the total quantity of labour available. On the other hand, no similar quantity can be considered as fixed in (II.4). This expresses the well-known property that any real system can give solutions only for relative prices but not for their absolute level. Therefore, the choice being arbitrary, we may put, for the time being, $P_n = \overline{P}_n$. Hence:

$$\text{(II.6)} \qquad \begin{cases} X_1 = a_{1n}\overline{X}_n, \\ X_2 = a_{2n}\overline{X}_n, \\ \quad \cdot \\ \quad \cdot \\ \quad \cdot \\ X_{n-1} = a_{n-1,n}\overline{X}_n ; \end{cases}$$

$$\text{(II.7)} \qquad \begin{cases} P_1 = a_{n1}\overline{P}_n, \\ P_2 = a_{n2}\overline{P}_n ; \\ \quad \cdot \\ \quad \cdot \\ \quad \cdot \\ P_{n-1} = a_{n,n-1}\overline{P}_n ; \end{cases}$$

which represent the solutions of the system for physical quantities and relative prices. The meaning of expressions (II.6) and (II.7) is fairly straightforward. Each of the coefficients $(a_{1n}, a_{2n}, \dots a_{n-1,n})$ expresses average *per-capita* demand for each commodity, so that (II.6) say that production of each commodity exclusively depends on demand. If there were no demand, there would be no production. On the other hand, each of the coefficients $(a_{n1}, a_{n2}, \dots a_{n,n-1})$ expresses the labour imput in each physical unit of output, so that (II.7) say that the price of each commodity is directly proportional to the quantity of labour required to produce it. In other words, prices, in this simple case, are explained by a pure labour theory of value.

3. *A necessary condition for full employment*

Condition (II.5) will recur time and again in the subsequent analysis and we may well investigate its economic meaning immediately.

From a mathematical point of view, the fulfilment of (II.5) is a necessary condition for each of the systems (II.3) and (II.4) to have positive solutions. However, non-fulfilment does not imply no solution. The coefficient matrix of (II.3)-(II.4) has a particular form (all its entries are zeros, except on the last row, on the last column, and along the diagonal), which means that the solutions of the systems can be derived directly, without substitution, from the first $(n-1)$ equations of (II.3) and from the first $(n-1)$ equations of (II.4) respectively. Therefore, relative prices and relative quantities are determined independently of condition (II.5), whose binding restrictions fall entirely on the last equation of each of the two systems. Let us see what this means. Suppose, for example, that $\sum_{i=1}^{n-1} a_{ni}\, a_{in} < 1$. This inequality implies two things. In the context of system (II.4) it implies [3] that $\Sigma a_{in} P_i < P_n$, namely that average *per-*

[3] From now on, for shortness sake, the limits of the summations will be omitted. In other words, all summations that will appear in our analysis have to be interpreted as running from 1 to $(n-1)$, unless otherwise specified.

capita expenditure is less than the income each worker receives: a situation of under-consumption. In the context of system (II.3) it implies that $\Sigma a_{ni}X_i < \overline{X}_n$, namely that total labour employed is less than total labour available: a situation of under-employment. Conversely, the fulfilment of (II.5) implies both full expenditure of total income and full employment.

Looking at condition (II.5) more analytically, we may notice that it is a sum of products $a_{ni}a_{in}$, $(i = 1, \ldots n-1)$, where each of these products has the property of being composed of one technical coefficient and of one consumption coefficient both referring to the same commodity. Therefore, each $a_{ni}a_{in}$ expresses the *proportion* of full employment national income which is spent in that sector (sector i), or — which is the same — the *proportion* of total labour which is employed in that sector. Evidently, the sum of these proportions must be equal to one in order to reach full employment. In those cases in which average *per-capita* expenditure turns out to be less than the income each worker receives, prices and quantities are determined all the same, but total production turns out to be less than potential production, and there will be unemployment in the system. The situation is expressed by condition (II.5) being under-satisfied, i.e. by the summation of (II.5) being less than unity. The gap between $\Sigma a_{ni}a_{in}$ and unity expresses the proportion of labour that remains unemployed.

Let me point out immediately the *macro-economic* nature of condition (II.5). It does not depend on the number of sectors that exist in the economy; it is just one condition referring to the economic system as a whole. We may express it by simply saying that there must be a total expenditure equal to potential national income if full employment is to be achieved.

When put in these terms, condition (II.5) sounds familiar. It expresses a conclusion which is very well known in economic theory, since J.M. KEYNES brought it out explicitly in his *General Theory*. The novelty here is that this macro-economic result emerges immediately from a pure production model of the type we are dealing with; a model, by the way, which has been developed on a multi-sector basis.

It must be added that, with the particular hypothesis of the present simple case (where no capital is needed) the condition that all income must be spent also means something more specific, namely that all income must be *consumed*. By adopting a Keynesian definition of savings, i.e. by defining savings as that part of personal incomes which is not spent, macro-economic condition (II.5) may also be expressed by saying that there can be *no net savings in the system,* as a necessary condition for full employment. Single individuals, of course, may save, but there must be other individuals who dissave by the same amount, so that savings and dissavings, on the whole, cancel each other out. The only form of savings which is possible for the system as a whole is in *real* terms, i.e. by carrying over durable (produced) commodities from one period to another. But within each period, total demand must be such as to induce the full utilization of the production potential, if full employment is to be reached. This is another important property of a production system. There must be demand — and in our simple case demand for consumption goods, whether durable or non-durables — for such an amount as to generate that quantity of production that requires the full employment of the existing labour force.

4. *Production by means of labour and capital*

We may now consider an economic system which has all the properties of the system analysed in the previous sections with one difference. The production processes, in order to be carried on, require another factor of production besides labour: capital. Each productive process includes, as in the previous case, a flow-input of labour and a flow-output of final commodity, but now it also requires a *stock* of capital, on which the labour services have to be performed in order to produce the final output.

Capital itself is a commodity, or rather a series of commodities, which have been produced in the economy, so that the *final* outputs of the system are now of two types: consumption goods, which are consumed as in the previous case, and capital goods, which are *invested* in the process of production. Capital goods are *durable* and therefore, once produced, they go to increase the existing stock of capital. However, although durable, they are not eternal. As a result of being used by labour they wear out, and must be continually re-integrated if production is to go on. This means that each productive process leading to a final commodity needs a further item. Besides a flow-input of labour and a stock of capital, each process also needs a flow-input of capital goods to keep the initial capital stock intact. This means that total production of capital goods, which represents total gross investment of the system, must be distinguished in two parts. A first part of it (replacement) simply goes to replace worn-out capital: in fact it is nothing but a cost of production, since it is needed in order that the productive processes may end up with the same amount of capital stock with which they began. The rest of the total production of capital goods, namely the excess over replacement, represents a net addition (*net investment*) to the existing capital stock.

But how is capital going to be measured? In our analysis, all commodities have been measured so far in two ways — in physical terms — whatever the physical unit may be — and in current price terms — physical quantities multiplied by prices. The same procedure will be used for capital goods, except that a particular physical unit will be adopted, namely *productive capacity*. Of course, a unit of productive capacity is, in an ordinary sense, a very composite physical commodity: not only does it include both what has been called « circulating » and what has been called « fixed » capital. It may actually be made up of a long series of different physical goods in different proportions and with different durability. The reason why such a unit of measure is here used is the same that prompted the use of the concept of a *final* commodity ([4]). Both concepts permit useful simplifications of exposition and will become especially helpful in the dynamic analysis which will follow in the next chapters.

5. *The physical stocks and flows of the system*

Let us consider, first of all, the physical aspect of our system in a given period of time. We are faced with a series of stocks and a series of flows.

At the beginning of the period, there exists a series of stocks of capital which have been inherited from previous periods. We may represent them by a vector

$$[K_1\ K_2, \ldots\ldots K_j, \ldots\ldots K_{n-1}]\ , \tag{II.8}$$

where each K_j stands for the stock of capital, measured in terms of productive capacity, in sector j ($j = 1, 2, \ldots n-1$). When our analysis begins, the K_j's are obviously given. However they are not given by « nature ». They are the result of production activity in earlier periods of time, each K_j being the sum of all net investments made in the past in sector j.

([4]) Conceptually, the notion of a unit of productive capacity and the notion of a final commodity have many similarities. Both a final commodity and a unit of capacity can, *at a given point in time*, be broken down into many distinct components: intermediate goods for the former, capital goods in an ordinary sense for the latter. These break-down relations, however, are valid only at a given point of time. When a movement through time is considered, the relations change - intermediate goods on one side and final goods on the other, capital goods on one side and productive capacity goods on the other, follow a path of their own. It is in connection with these time movements that final goods and productive capacity goods will become particularly useful in the subsequent dynamic analysis.

We might say that there is also another stock in the system, at the beginning of the period we are considering, namely population. This stock, however, for economic purposes, is not relevant as such, i.e. *as a stock* (except in a slave society, which is outside our interests). Its economic relevance is connected only with the *flows* it calls into being.

Let us come, therefore, to consider these flows. As in the previous case, all the members of the community are grouped in a final sector n. This final sector provides the labour services to all productive processes and owns the stocks of capital. Moreover, it exerts the demand for all the final goods, which are now of two types — consumption goods and investment goods.

The flows of consumption goods, as before, come from $(n-1)$ consumption goods sectors. In addition to these, we have now a whole series of flows of investment goods, which will be denoted with the suffix k. As a first step to the more general formulation of section 7, we shall assume that capital is required only for the production of consumption goods, while capital goods can be produced exclusively from labour. Since the unit of measure of capital is different for each consumption good sector, the process for the production of capital goods must itself be expanded into $(n-1)$ sectors. Our system thereby acquires $(n-1)$ new variables: X_{k_1}, X_{k_2}, ..., $X_{k_{n-1}}$, each of which represents the production of capital for the corresponding consumption goods sector. This production of capital goods is devoted partly to new investment (expressed by demand coefficients in the final sector: $a_{k_1 n}$, $a_{k_2 n}$, ..., $a_{k_{n-1} n}$) and partly to replacement of worn-out capacity (expressed by replacement coefficients: $a_{k_1 1}$, $a_{k_2 2}$, ..., $a_{k_{n-1} n-1}$). Hence the whole structure of physical flows can be represented by the following system of equations, which takes the place of (II.3):

$$
\text{(II.9)}\qquad
\begin{bmatrix}
-1 & & & & & & & a_{1n} \\
& -1 & & & & & & a_{2n} \\
& & \ddots & & & & & \vdots \\
a_{k_1 1} & & & -1 & & & & a_{n-1,n} \\
& a_{k_2 2} & & & -1 & & & a_{k_1 n} \\
& & \ddots & & & \ddots & & \vdots \\
& & & a_{k_{n-1}\, n-1} & & & -1 & a_{k_{n-1} n} \\
a_{n1} & a_{n2} & \cdots & a_{n,n-1} & a_{nk_1} & a_{nk_2} \;\cdot & a_{nk_{n-1}} & -1
\end{bmatrix}
\begin{bmatrix}
X_1 \\ X_2 \\ \vdots \\ X_{n-1} \\ X_{k_1} \\ \vdots \\ X_{k_{n-1}} \\ X_n
\end{bmatrix}
=
\begin{bmatrix}
0 \\ 0 \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ 0 \\ 0
\end{bmatrix}
$$

A few words may be added about the replacement coefficients. Each $a_{k_i i}$ $(i=1, 2, \ldots, n-1)$ represents the amount of capital, in physical terms, which is required in each period of time in order to keep productive capacity intact. This is the general definition of the $a_{k_i i}$'s. However, it may be useful to adopt a simplification, because of its immediately apparent meaning. We may assume that in each period of time and in each sector, a *constant* proportion of the productive capacity vanishes owing to wear and tear, so that $T_i = 1/a_{k_i i}$ $(i=1, 2, \ldots n-1)$ represents a quantity which is very near (though, depending on the rate of growth of the sector, not exactly coincidental with) the average life-time of physical capital goods in sector i.

By now following exactly the same steps as in section 2, the condition that must be satisfied in order that the system may have non-trivial solutions emerges as

$$
\text{(II.10)}\qquad \sum a_{ni}\, a_{in} + \sum \frac{1}{T_i} a_{nk_i}\, a_{in} + \sum a_{nk_i}\, a_{k_i n} = 1 \,,
$$

and the solutions for the physical quantities take the form

$$
\text{(II.11)}\quad
\begin{cases}
X_1 = a_{1n}\overline{X}_n\,, \\
\quad\vdots \\
X_{n-1} = a_{n-1,n}\overline{X}_n\,, \\
\quad\vdots \\
X_{k_1} = \left(a_{k_1 n} + \dfrac{1}{T_1}\,a_{1n}\right)\overline{X}_n\,, \\
\quad\vdots \\
X_{k_{n-1}} = \left(a_{k_{n-1} n} + \dfrac{1}{T_{n-1}}\,a_{n-1,n}\right)\overline{X}_n\,.
\end{cases}
$$

As can be seen, the first $(n - 1)$ solutions, referring to consumption goods, are again of the simple type of the previous case: they actually coincide with the (II.6). But the following $(n - 1)$ solutions, referring to investment goods, now contain two elements, expressing the fact that production of these goods is generated by two distinct types of demand: demand for new investments, and demand for replacement of worn-out capacity.

6. *The structure of prices*

From the physical flows which have just been considered, another system of equations can be derived in terms of prices. At first, one might think that one should write the same matrix of (II.9) (but transposed) and multiply it by prices. This procedure, however, is no longer sufficient because the production processes now require capital, besides labour, and therefore the total income which flows to sector n has two components: a remuneration for labour (wages) and a remuneration for capital (profits). In each sector, the value of total production must be equal to the amount of wages due the labourers plus the amount of profits due the owners of the capital which is used in the production processes (in terms of productive capacity: $X_1, X_2, \ldots, X_{n-1}$). By calling W the wage rate and π the rate of profit, system (II.4) becomes:

(II.12)

$$
\begin{bmatrix}
-1 & & & & & & & & a_{k_1 1} & & & a_{n1} \\
& -1 & & & & & & & & a_{k_2 2} & & a_{n2} \\
& & \cdot & & & & & & & \cdot & & \cdot \\
& & & \cdot & & & & & & & \cdot & \cdot \\
& & & & \cdot & & & & & & \cdot & \cdot \\
& & & & & -1 & & & & & a_{k_{n-1} n-1} & a_{n,n-1} \\
& & & & & & -1 & & & & & a_{nk_1} \\
& & & & & & & \cdot & & & & \cdot \\
& & & & & & & & \cdot & & & \cdot \\
& & & & & & & & & \cdot & & \cdot \\
& & & & & & & & & & -1 & a_{nk_{n-1}} \\
a_{1n} & a_{2n} & \cdots & & a_{n-1,n} & a_{k_1 n} & a_{k_2 n} & \cdots & & & a_{k_{n-1} n} & -1
\end{bmatrix}
\begin{bmatrix}
P_1 \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ P_{n-1} \\ P_{k_1} \\ \cdot \\ \cdot \\ \cdot \\ P_{k_{n-1}} \\ W
\end{bmatrix}
+
\begin{bmatrix}
\frac{\pi X_1 P_{k_1}}{X_1} \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \frac{\pi X_{n-1} P_{k_{n-1}}}{X_{n-1}} \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \\ \frac{\pi}{X_n} \Sigma X_i P_{k_i}
\end{bmatrix}
=
\begin{bmatrix}
0 \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ 0 \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ 0
\end{bmatrix}
$$

Here we are faced with a complication. Comparing (II.12) with system (II.4) we can see that the variable P_n has been eliminated, but only in order to be replaced by *two* new variables: W and π. This means that, if we want to keep definite solutions for relative prices, we must assume one of the variables as given, or alternatively we must introduce a new relation.

This is an old dilemma in economic analysis. When the classical economists (RICARDO in particular) faced it, they thought they could take the wage rate as given from outside the sphere of economics. Later on, the marginalist economists chose the other alternative. As is well known, they introduced a series of new hypotheses, which amounted to treating capital goods *as if* they were natural resources, and built up a whole theory behind the technical coefficients — the theory of marginal productivity.

In the present work, I am in a sense going back to the Classics' approach, but with a reversal of their hypothesis. At this stage, I shall take as given the rate of profit (π). It must be made clear, however, that this step does not have, here, the same meaning that taking a given wage rate had for the Classics. The rate of profit is not an exogenous magnitude in the present theoretical scheme (as technology and consumers' preferences are). What this taking the rate of profit as given here means is a simple assertion that the rate of profit is not determined by the elements of the model so far considered. Of course, the rate of profit is determined by elements which come within the scope of economic investigation and which must be examined. We shall come back to this problem later on in chapter V. For the time being, the only anticipation of that discussion needed here is that the rate of profit is bound to remain roughly constant, or rather, to show a roughly constant trend through time.

This result is relevant here because it allows us to keep the rate of profit among the constants of our analysis. Thereby we are in a position to express the new vector appearing in (II.12) in terms of prices, and to incorporate its constant elements into the coefficient matrix. The rate of profit itself need not necessarily be exactly the same in all sectors. There may be particular elements of risk and uncertainty in each sector, to be added to a basic rate of profit. Thus system (II.12) becomes

(II.13)

$$
\begin{bmatrix}
-1 & & & \left(\pi_1+\frac{1}{T_1}\right) & & & & a_{n1} \\
 & \cdot & & & \cdot & & & \cdot \\
 & & \cdot & & & \cdot & & \cdot \\
 & & & \cdot & & & \cdot & \cdot \\
 & & & -1 & & \left(\pi_{n-1}+\frac{1}{T_{n-1}}\right) & & a_{n,n-1} \\
 & & & & -1 & & & a_{nk_1} \\
 & & & & & \cdot & & \cdot \\
 & & & & & & \cdot & \cdot \\
 & & & & & & \cdot & \cdot \\
 & & & & & & -1 & a_{nk_{n-1}} \\
a_{1n} & \cdots & a_{n-1,n} & (a_{k_1 n}-\pi_1 a_{1n}) & \cdots & (a_{k_{n-1}n}-\pi_{n-1} a_{n-1,n}) & & -1
\end{bmatrix}
\begin{bmatrix} P_1 \\ \cdot \\ \cdot \\ \cdot \\ P_{n-1} \\ P_{k_1} \\ \cdot \\ \cdot \\ \cdot \\ P_{k_{n-1}} \\ W \end{bmatrix}
=
\begin{bmatrix} 0 \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ 0 \\ 0 \end{bmatrix}
$$

Here again we have obtained a system which is linear and homogenous. The coefficient matrix looks different from the matrix of (II.9). However — interestingly enough — the determinants of the two matrices are exactly the same, as can be easily demonstrated (5). The condition for non-trivial so-

(5) The determinant of the matrix of (II.9), after substituting the $\frac{1}{T_i}$'s for the $a_{k_i i}$'s comes out as:

$$\sum a_{ni}\, a_{in} + \sum a_{nk_i}\, a_{k_i n} + \sum \frac{1}{T_i}\, a_{nk_i}\, a_{in} - 1 \ ,$$

and the determinant of the matrix of (II.13) as:

$$\sum a_{ni}\, a_{in} + \sum (a_{k_i n} - \pi_i a_{in})\, a_{nk_i} + \sum \left(\pi_i + \frac{1}{T_i}\right) a_{in}\, a_{nk_i} - 1 \ .$$

This second expression clearly reduces to the first by expansion. We obtain:

$$\sum a_{ni}\, a_{in} + \sum a_{nk_i}\, a_{k_i n} + \sum \frac{1}{T_i}\, a_{in}\, a_{nk_i} - \sum \pi_i\, a_{in}\, a_{nk_i} + \sum \pi_i\, a_{in}\, a_{nk_i} - 1 \ ,$$

where, as can be seen, the last two summations cancel out.

lutions of (II.13) is therefore exactly the same as that which has been found already for (II.9), namely (II.10), and the solutions for prices (relative prices) come out as:

$$
\text{(II.14)} \quad \begin{cases} P_1 = \left[a_{n1} + \left(\pi_1 + \frac{1}{T_1}\right)a_{nk_1}\right] W , \\ \cdot \\ \cdot \\ \cdot \\ P_{n-1} = \left[a_{n,n-1} + \left(\pi_{n-1} + \frac{1}{T_{n-1}}\right) a_{nk_{n-1}}\right] W , \\ P_{k_1} = a_{nk_1} W , \\ \cdot \\ \cdot \\ \cdot \\ P_{k_{n-1}} = a_{nk_{n-1}} W . \end{cases}
$$

As the reader can see, the last $(n - 1)$ prices are still of the simple type of the previous section. However, this is only because of the simplifying assumption that capital goods require no capital goods to be produced. The formulation for the $(n - 1)$ prices of consumption goods are more general and more interesting. Each price is expressed as a sum of two elements: the prime cost (a_{ni} = quantity of labour required to produce a unit of commodity) and the gross profit mark-up $(\pi_i + \frac{1}{T_i})a_{nk_i}$, which in turn is composed of the rate of profit in sector i (π_i), and of the depreciation allowance ($1/T_i$), both of them being proportional to the capital intensity of the productive process (a_{nk_i} = quantity of labour required to produce one unit of productive capacity).

Already at this stage, a pure labour theory of value is no longer valid. The only case in which it still stands becomes a very peculiar one: the case in which either the rate of profit is zero or the proportion of replacements (i.e. of indirect labour) to direct labour is exactly the same in all sectors. In general, therefore, relative prices will depend both on the sum of direct and indirect labour required, and on the proportion between the two, i.e. on the capital intensity of the production processess. However, it is important to notice that our approach has made it possible — as the (II.14) show — to express the capital components of prices in terms which are directly and unambiguously comparable to labour. These capital components may thus be added to direct and indirect labour. In the (II.14) they are added, and the total sum is multiplied by the wage rate. We may, therefore, conclude that formulae (II.14) express a theory of value which is indeed no longer in terms of pure labour but is — as we may put it — in terms of *labour equivalents*. In the case considered here, the amount of labour equivalent corresponding to the employment of capital is a fraction (represented by the rate of profit), of the amount of labour required to produce capital goods.

7. *A more complex case involving capital for the production of capital*

The foregoing analysis has been based on the simplifying assumption that capital goods are not needed for the production of capital goods, but the simplification has been made only in order to keep the formulation as short as possible.

Dropping this assumption does not entail any conceptual difficulty, it only requires a few more algebraical manipulations. Assume, for example, in order not to go *and infinitum*, that each of the capital goods sectors makes capital goods for itself and for the corresponding consumption goods sector, and denote by γ_i, in each sector i, $(i = 1, 2, \ldots, n-1)$, the ratio of one unit of capital goods expressed in terms of capacity for the consumption goods sector to one unit of capital goods expressed in terms of capacity for the capital goods sector. Then a new series of $(n-1)$ replacement coefficients for the capital goods sectors have to be introduced in matrix (II.9), in addition to those already considered in the consumption goods sectors. Similarly a new series of $(n-1)$ profit and depreciation mark-ups, for the prices of investment goods, has to be introduced in matrix (II.13) in addition to those already considered for the prices of consumption goods.

In both matrices the new $(n-1)$ elements fall along the second half of the main diagonal [6]. The solutions, as can be easily seen, are again very simple to reach and come out as follows:

$$\text{(II.15)} \quad \begin{cases} X_1 = a_{1n} \overline{X}_n , \\ \quad \vdots \\ X_{n-1} = a_{n-1,n} \overline{X}_n , \\ X_{k_1} = \dfrac{T_{k_1}}{T_{k_1} - \gamma_1} \left(a_{k_1 n} + \dfrac{1}{T_1} a_{1n} \right) \overline{X}_n , \\ \quad \vdots \\ X_{k_{n-1}} = \dfrac{T_{k_{n-1}}}{T_{k_{n-1}} - \gamma_{n-1}} \left(a_{k_{n-1} n} + \dfrac{1}{T_{n-1}} a_{n-1,n} \right) \overline{X}_n , \end{cases}$$

(6) The condition under which the matrices of this more complex model are singular emerges as follows:

$$\sum a_{ni} a_{in} + \sum \frac{T_{k_i}}{T_{k_i} - \gamma_i} a_{nk_i} a_{k_i n} + \sum \frac{1}{T_i} \frac{T_{k_i}}{T_{k_i} - \gamma_i} a_{nk_i} a_{in} = 1 .$$

$$
\text{(II.16)} \quad \left\{
\begin{aligned}
P_1 &= \left[\left(\pi_1 + \frac{1}{T_1}\right)\left(\frac{T_{k_1}}{T_{k_1} - \pi_{k_1} T_{k_1} - \gamma_1}\right) a_{nk_1} + a_{n1}\right] W , \\
&\;\vdots \\
P_{n-1} &= \left[\left(\pi_{n-1} + \frac{1}{T_{n-1}}\right)\left(\frac{T_{k_{n-1}}}{T_{k_{n-1}} - \pi_{k_{n-1}} T_{k_{n-1}} - \gamma_{n-1}}\right) a_{nk_{n-1}} + a_{n,n-1}\right] W , \\
P_{k_1} &= \frac{T_{k_1}}{T_{k_1} - \pi_{k_1} T_{k_1} - \gamma_1} a_{nk_1} W , \\
&\;\vdots \\
P_{k_{n-1}} &= \frac{T_{k_{n-1}}}{T_{k_{n-1}} - \pi_{k_{n-1}} T_{k_{n-1}} - \gamma_{n-1}} a_{nk_{n-1}} W .
\end{aligned}
\right.
$$

These two new series of solutions are basically similar to (II.11) and (II.14), although they are a little more complicated. Each production of capital goods now covers not only replacement and new investment for consumption goods sectors, but also replacement and new investment for capital goods sectors. Similarly each of the prices now covers not only the profit and depreciation allowance for consumption goods sectors but also the profit and depreciation alowance for the corresponding capital goods sector.

Since, in this case, the productive processes of capital require as input a part of their own outputs, there is one more necessary condition for *positive* solutions — or rather a series of necessary conditions — explicitly brought out by (II.15) - (II.16). Algebraically, the conditions are

$$
\text{(II.17)} \qquad T_{k_i} > \gamma_i , \qquad i = 1, 2, \ldots, n-1 ;
$$

meaning that the total output from the employment of one machine, in the whole course of its life, must be more than one machine of the same type. If it were not so (i.e. if the processes to produce capital goods required more as inputs than what they give as outputs) production would be impossible. The reader will also notice that the theory of value implied by (II.16) is analogous to the theory implied by (II.14). All components of prices are expressed in terms of labour and labour equivalents.

These similarities will be very useful in the subsequent analysis, where it does not make any difference whether the (II.11), (II.14) or the (II.15), (II.16) are used, except in some particular cases. Normally, therefore, we shall be able, without lack of generality, to carry on our analysis in terms of the (II.11), (II.14), which are much simpler. For analogous reasons of simplicity, the procedure will normally be followed of using a single rate of profit (π) and the same replacement coefficient ($\frac{1}{T}$) for all the sectors.

8. *The conditions for equilibrium*

We have not yet commented on the new form taken by the full employment condition. Expression (II.10) represents, in mathematical terms, the condition that must be satisfied in order that the linear and homogeneous systems (II.9) and (II.13) may have non-trivial solutions. It clearly takes the place of (II.5) in the previous case and has exactly the same economic meaning: a necessary condition for reaching full employment. Unlike (II.5), expression (II.10) now makes a distinction among three different types of demand: demand for consumption goods, demand for new investments and demand for replacing worn-out capital goods. However, the importance of this distinction is only to be seen in the dynamic analysis which will follow — a different composition of demand in a certain period of time evidently has different consequences on the stocks of capital goods in the ensuing period. But for the time being, and as far as a short-run analysis is concerned, the composition of demand does not matter. To fulfil condition (II.10), that is to reach full employment, the only requirement that is needed is that the *sum* of all types of demand be such as to imply a total expenditure equal to total potential income.

What must be added, however, is that the fulfilment of (II.10) is no longer enough, because it only refers to the flow aspect of the system, and thereby implies full employment only potentially. Some other conditions must be fulfilled with regard to the stocks. First of all, there must be enough productive capacity in the system to make it possible to produce what is required by potential demand, i.e.

$$\text{(II.18)} \qquad K_i \geq X_i \,, \qquad i = 1, 2, \dots (n-1) \,.$$

Yet this could not be defined a satisfactory situation, if the capacities were far beyond what is required by a full employment demand. We have to impose also the conditions of full capacity utilization, namely:

$$\text{(II.19)} \qquad K_i \leq X_i \,, \qquad i = 1, 2, \dots (n-1) \,.$$

It follows that, for the (II.18) and (II.19) to be simultaneously satisfied, all the K_i's must be equal to the X_i's.

To sum up, two types of conditions are now necessary for the systems (II.9) and (II.13) to hold, namely

$$\text{(II.20)} \qquad \sum a_{ni}\, a_{in} + \sum a_{nk_i}\, a_{k_i n} + \sum \frac{1}{T_i}\, a_{nk_i}\, a_{in} = 1 \,,$$

and

$$\text{(II.21)} \qquad K_i = X_i \,, \qquad i = 1, 2, \dots (n-1) \,.$$

We may notice that (II.20), like (II.5), is a macro-economic condition: the whole national income must be spent, if full employment is to be reached. On the other hand, (II.21) represents *a series* of sectoral conditions: there is one condition for each sector. Each sector i must be endowed with that stock of productive capacity which is necessary to produce the amount of commodity i which is demanded.

At this point, one may wonder what would happen if (II.20)-(II.21) were not satisfied. It clearly depends on how the non-fulfilment comes about. A few interesting cases may perhaps be usefully considered. Inequalities of type (II.18) plus the left hand side of (II.20) being lower than its right hand side mean idle capacity and less than full empoyment: a situation which we may call one of Keynesian under-employment (7). On the other hand, inequalities of type (II.19) again plus the left-hand side of (II.20) lower than its right hand side correspond to a situation in which the capital structure is smaller than the one which would make full employment possible: a situation which may be called one of Marxian under-employment. The two opposite cases as to condition (II.20) represent situations of inflation of different types, due respectively to lack of labour and to lack of capital.

A further question that may arise at this point is the following: if any of these cases, or if any other situation takes place, in which (II.20) and (II.21) are not satisfied, how is the system

(7) Of course, KEYNES had a *behavioural* theory about this situation, with reference to a capitalist economic system. In terms of the present model, KEYNES' theory was that, for psychological reasons,

$$\Sigma a_{ni}\, a_{in} < 1,$$

namely that total demand for consumer goods on the whole tends to be smaller than full employment income. Of course demand for investment goods might be such a proportion of total income as to make up for the difference to unity in the inequality stated above. But KEYNES pointed out that there is no reason to necessarily expect this, because the two types of demand depend on different factors. As is well known, he thought that in fact the most likely situation to arise is one in which there is lack of effective demand and excess of productive capacity.

going to react? The answer depends on the particular institutions that the system has adopted. But, as said already, it is not the purpose of the present enquiry to introduce any particular theory corresponding to any specific institutional set-up. The purpose is simply to find and to specify the conditions that *must* be satisfied in any case, if full employment is to be reached.

To conclude, when both (II.20) and (II.21) are fulfilled, then the two systems of equations (II.9) and (II.13) entirely hold. This means that the economic system they represent is in a *situation* which we may call one *of equilibrium*, an expression simply taken to mean full employment of labour and full utilization of productive capacity.

9. *Towards a dynamic analysis*

The foregoing analysis, after introducing capital, has acquired an important characteristic: although still a short-term analysis it is not a static analysis. Even if we suppose that the equilibrium conditions (II.20) - (II.21) are satisfied, we cannot say that we are at the end of our enquiry. These conditions refer to a certain period of time, but just in order to be fulfilled in that period, they may contain some elements (investments) whose mere existence means modifying in the following period those magnitudes (stocks of capital) on which the previous equilibrium was based. In this theoretical framework, therefore (unlike what happened in the traditional type of economic enquiries), the attainment of the situation of equilibrium in a given period of time does not mean at all that all problems have been solved.

We are inevitably led, by our own analysis, from the investigation of equilibrium conditions in a given period of time, to the investigation of equilibrium conditions over time.

CHAPTER III

THE SIMPLEST CASE OF ECONOMIC EXPANSION - POPULATION GROWTH WITH CONSTANT RETURNS TO SCALE

1. *A simple dynamic model*

Our inquiry will now venture into what happens *after* the single period so far considered has elapsed, and into what happens in general as time goes by. From an analytical point of view, there are two notions of time which may be adopted. Time may be conceived of as a succession of finite periods, with the supposition that changes take place only between one period and the other. Or, time may be conceived of as bringing along changes in a continuous way, so that the finite periods of the previous procedure become so short as to be infinitesimal. For our purposes, it is irrelevant whether the first or the second procedure is adopted. But since the second procedure makes things simpler, from our point of view, it will normally be followed all through the following dynamic analysis. However, the arguments will also be re-cast now and then in terms of the first procedure, when that appears useful and illuminating.

If we look beyond the single period of time, all quantities considered so far must be dated. The previous short-run model remains valid within each period, but from one period to another those quantities which have hitherto been taken as given (population, technical knowledge, consumption patterns, capital stocks) may undergo important changes.

We shall consider in this chapter the simplest of all cases of dynamic change. We shall assume that, as time goes on, the only « exogenous » factor which is moving is population. This simple case has been extensively dealt with already in current economic literature, and the purpose of the present chapter is not, therefore, to obtain new results. The purpose is simply to evince the connections with the known growth models and at the same time to develop formulations which will be needed in the following analysis. Our assumptions may be listed schematically as follows:

a) first of all, the *initial conditions* are such that, when our analysis begins, at a time defined as zero, the system is operating in equilibrium. There is full employment of labour and full utilization of productive capacity;

b) as time goes on, population increases at a steady percentage rate g, so that

$$X_n(t) = X_n(0)\, e^{gt}\,, \tag{III.1}$$

where t denotes time, and e is the well-known base of the natural logarithms;

c) technical conditions remain fixed over time; expansion takes place at constant returns to scale. In other words, all technical coefficients (the a_{ni}'s, the a_{nk_i}'s and the T_i's, $i = 1, 2, \ldots n-1$) remain unchanged in time;

d) consumers' tastes also remain constant, which means that, if individuals continue to receive the same income, their consumption — i.e. all the coefficients a_{in}'s, $i = 1, 2, \ldots (n-1)$ — remain constant through time.

These are all the assumptions that are needed, besides of course the convention (discussed in section 6 of the previous chapter) of taking the rate of profit as given. The reader may have noticed that there is a series of coefficients which have not been mentioned: the $a_{k_i n}$'s, namely the demand coefficients for new investments. The reason is that these coefficients cannot be specified in advance. They are themselves unknowns, in the present model, as they must be determined in such a way as to be consistent with the growing productive potential of the system.

Our task is, first of all, to find out the conditions under which equilibrium may be maintained over time; and then to investigate the time-paths of the variables of the system.

2. *The conditions for a dynamic equilibrium*

We might begin by saying that, in order to maintain equilibrium over time, conditions (II.20) and (II.21), which are satisfied by hypothesis at time zero, must also and constantly be kept satisfied as time goes on. A statement of this kind, however, now becomes uninteresting and artificial, because those two conditions cease to be independent of one another, when time is allowed to elapse.

If population is growing, both the supply of labour services and the demand for each product increase, as time goes on. In order that the new labour force may find employment and the growing potential demand may become effective, productive capacity must also increase. Thus, the equilibrium conditions for productive capacity in time become

$$\text{(III.2)} \qquad \frac{d}{dt}[K_i(t)] = \frac{d}{dt}[X_i(t)]\,, \quad i = 1,\ 2,\ \ldots\ (n-1)\ ;$$

which simply means that there must be an increase in the productive capacity of each commodity, parallel to the increase in its demand.

The formulation may be expressed in a more helpful way by making use of a few definitions. We know from (II.11) that each production of capital goods (X_{k_i}) is composed of two parts: replacement of worn-out capacity ($\frac{1}{T_i} a_{in} X_n$); and new investment ($a_{k_i n} X_n$). Denoting these two parts respectively by X'_{k_i} and X''_{k_i}, so that

$$\text{(III.3)} \qquad X_{k_i}(t) \equiv X'_{k_i}(t) + X''_{k_i}(t) ,$$

we have

$$\text{(III.4)} \qquad X''_{k_i}(t) \equiv \frac{d}{dt} [K_i(t)] , \qquad i = 1, 2, \ldots (n-1).$$

By substituting now (III.4) into (III.2), we obtain

$$\text{(III.5)} \qquad X''_{k_i}(t) = \frac{d}{dt} [X_i(t)] , \qquad i = 1, 2, \ldots (n-1),$$

which amounts to expressing the capacity equilibrium conditions in the form of equilibrium relations between the production of capital goods in each period of time and the *rate of change* in that period of the corresponding consumption goods. Using the (II.11) and (III.1), the (III.5) become

$$a_{k_i n} X_n(0)e^{gt} = \frac{d}{dt} [a_{in} X_n(0)e^{gt}] , \quad i = 1, 2, \ldots (n-1),$$

or

$$a_{k_i n} X_n(0)e^{gt} = g \, a_{in} X_n(0)e^{gt} ,$$

and finally

$$\text{(III.6)} \qquad a_{k_i n} = g \, a_{in} , \qquad i = 1, 2, \ldots (n-1).$$

This is a formulation of real interest. It states the series of sectoral productive capacity conditions in terms of flows (and no longer in terms of stocks) by specifying a very definite relation which must hold between each demand coefficient for new investments and the corresponding demand coefficient for consumption goods. It states that, as a condition to endow the system with the equilibrium productive capacity, each sectoral net investment, in physical terms, must be equal to the corresponding sectoral final demand multiplied by the rate of growth of population. This determines sectoral equilibrium net investments in the whole economic system:

$$\text{(III.7)} \qquad X''_{k_i}(t) = g\, a_{in} X_n(t) \qquad i = 1, 2, \dots (n-1).$$

These conditions may, therefore, be called *the capital accumulation conditions* for keeping full employment over time. If we like, they may be expressed also in terms of ratios between quantities evaluated at current prices. After multiplying both sides of (III.7) by P_{k_i} and dividing by $P_i X_i$, we obtain

$$\text{(III.8)} \qquad \frac{P_{k_i} X''_{k_i}}{P_i X_i} = g \frac{P_{k_i} X_i}{P_i X_i}, \qquad i = 1, 2, \dots (n-1),$$

which mean that, in equilibrium, each sectoral ratio of investment to production, evaluated at current prices, is equal to the percentage rate of population growth multiplied by the corresponding sectoral capital-output ratio.

When the capital accumulation conditions are kept satisfied, the system — as time goes on — is being constantly endowed with exactly those stocks of productive capacity which are necessary in order to provide full employment for all the workers. This, however, does not yet necessarily imply full employment and full utilization of productive capacity. In order that equilibrium be reached, the other condition — the macro-economic relation referring to the system as a whole — must also be satisfied. This condition, after substituting (III.6) into (II.20), becomes

$$\sum a_{ni}\, a_{in} + \left(g + \frac{1}{T}\right) \sum a_{nk_i}\, a_{in} = 1 \,. \qquad \text{(III.9)}$$

The economic meaning is again that there must be a total expenditure equal to total potential gross income if full employment and full capacity utilization are to be reached. However, (III.9) now also requires a very definite division of the total expenditure between new investments, replacements, and consumption. The effect of substituting into it the capital accumulation conditions has been to specify the magnitude of the term $(g + \frac{1}{T}) \sum a_{nk_i}\, a_{in}$, which is nothing but an analytical break-down of the equilibrium ratio of total gross investments to gross income in the system as a whole. This equilibrium ratio, as can be seen, is exclusively determined by the three exogenous factors of our model: population growth, technology and consumers' preferences. Therefore, if (III.9) is to be satisfied, it is the total effective demand for consumption goods that must absorb the whole remaining proportion — represented by the first addendum of (III.9) — of potential gross income. Condition (III.9), since it thereby determines the size of total effective demand, may be called *the effective demand condition* for keeping full employment.

We may notice that this condition, as it stands, presents no problem *through time*. Since all coefficients are constant, once (III.9) is satisfied at time zero, it will remain satisfied all time.

We may also look at it in another and more familiar way, by re-writing it as

(III.10) $$1 - \sum a_{ni}\, a_{in} - \frac{1}{T} \sum a_{nk_i}\, a_{in} = g \sum a_{nk_i}\, a_{in}\,.$$

The left hand side expresses the proportion of total gross income which is not spent either on consumption goods or on capital replacement (the aggregate ratio of net savings to gross income), and the right hand side expresses the equilibrium aggregate ratio of net investments to gross income. Now, the right hand side, by utilizing conditions (III.6), expresses the equilibrium investment-income ratio as a multiplication of the rate of population growth by what can be shown to be, after a few algebraical re-arrangements [1], the over-all capital-output

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[1] The proof may be given in the following way.
Call $Y = \Sigma\, X_i P_i + \Sigma\, X''_{k_i} P_{k_i}$ and multiply both sides of (III.9) by $\frac{X_n W}{Y}$:

$$\frac{X_n W}{Y} \sum a_{ni}\, a_{in} + \left(g + \frac{1}{T}\right) \frac{X_n W}{Y} \sum a_{nk_i}\, a_{in} = \frac{X_n W}{Y}\,.$$

Add now to both sides the ratio of equilibrium total profit to national income:

$$\frac{1}{Y} \sum \pi P_{k_i} X_i + \frac{X_n W}{Y} \sum a_{ni}\, a_{in} + \left(g + \frac{1}{T}\right) \frac{X_n W}{Y} \sum a_{nk_i}\, a_{in} = $$
$$= \frac{X_n W}{Y} + \frac{1}{Y} \sum \pi P_{k_i} X_i\,.$$

We obtain

$$\frac{1}{Y} \sum \pi a_{nk_i}\, a_{in} W X_n + \frac{X_n W}{Y} \sum a_{ni}\, a_{in} + \left(g + \frac{1}{T}\right) \frac{X_n W}{Y} \sum a_{nk_i}\, a_{in} =$$
$$= \frac{X_n W + \sum \pi P_{k_i} X_i}{Y}\,,$$

$$\frac{1}{Y} \sum a_{in} X_n \left[a_{ni} + \left(\pi + \frac{1}{T}\right) a_{nk_i}\right] W + g \frac{1}{Y} \sum a_{nk_i} W\, a_{in} X_n = \frac{Y}{Y}\,,$$

ratio. This is a well-known relation. As the reader will realize, condition (III.10) is simply a more analytical formulation (referred to the case of population growth) of what has become generally known in macro-economic theory as the HARROD-DOMAR equation (²).

To conclude, two types of conditions must necessarily be satisfied in order to keep equilibrium over time. There is, first of all, a series of capital accumulation condition (III.6), ensuring that each sector be endowed all the time with the additional productive capacity required by the expanding demand. These conditions state that, in each sector, the ratio of new investments to the level of production must be equal to the technologically determined capital-output ratio multiplied by the rate of population growth. Secondly, in order to ensure the full utilization of the productive capacity that thereby comes into being and of the available labour force, the macro-economic effective demand condition (III.9) must also be satisfied. This condition determines the equilibrium division of total expenditure between consumption, replacements, and new investments. It states that, given the total amount of equilibrium investments by the first series of conditions and by the replacement requirements, total demand for consumption goods must be such as to absorb the whole remaining part of potential gross income.

$$\frac{1}{Y}\sum X_i P_i + g\frac{1}{Y}\sum X_i P_{k_i} = 1 .$$

And since, in equilibrium, $X_i = K_i$, we obtain:

$$1 - \frac{\sum X_i P_i}{Y} = g\frac{\sum K_i P_{k_i}}{Y} . \qquad Q.E.D.$$

(²) See R. F. HARROD, *An Essay in Dynamic Theory* in « The Economic Journal » 1939, and *Towards a Dynamic Economics*, London 1948; E. DOMAR, *Capital Expansion, Rate of Growth and Employment*, in « Econometrica », 1946.

3. *A more complete formulation of the effective demand condition*

It may be useful to take a further step, at this point, to complete formulation (III.9). So far, population has been referred to in general terms, both with respect to demand of commodities and with respect to supply of labour. This means assuming implicitly that population and labour force are the same thing. But in practice, this is not so. All people do contribute to the demand of commodities, but only a fraction of them (representing the active population) actually take part in the process of production. Moreover, a further complication arises in connection with the fact that our coefficients have a time dimension. The point may be made clearer by supposing a finite period, for example a year, as the unit of time. In this case, the consumption coefficients refer to yearly *per-capita* consumption; but, as a normal practice, the technical coefficients are referred to that fraction of the unit of time which corresponds to actual working time. If this is done, the a_{in}'s and the a_{ni}'s come to be no longer expressed in terms of the same number of people and not even to refer to the same unit of time.

To be consistent, a correction must be made on the technical coefficients and the simplest way to do it is to divide each of them by two parameters, the first of which — let us call it α — representing the proportion of active to total population, and the second — let use call it β — representing the proportion of, let us say, working hours to the total number of hours forming the unit of time considered. Evidently:

$$1 > \alpha > 0 ,$$
$$1 > \beta > 0 .$$

These corrections do not affect the expressions we found for prices — for example the (II.14) — where the technical coefficients appear, provided that the wage rate is also referred to a unit of actual working time. They do affect, on the other hand, the expression of the effective demand condition for equilibrium. (This is after all intuitive: the conditions for reaching full employment are evidently different according to the ratio of active to total population and to the length of the working week). Hence, after introducing α and β, (III.9) becomes

$$\text{(III.11)} \qquad \frac{1}{\alpha\beta} \sum a_{ni} a_{in} + \left(g + \frac{1}{T} \right) \frac{1}{\alpha\beta} \sum a_{nk_i} \, a_{in} = 1 \, ,$$

which must be considered as a more complete formulation of the effective demand condition for a dynamic equilibrium.

4. *The dynamic movements of relative prices, physical quantities and other economic variables*

To find now how prices and physical quantities move as time goes by is a very easy task. As to prices, it can immediately be seen from the (II.14) or from the (II.16) that, under the present hypotheses, all their components are constant in time, so that *all relative prices remain constant as time goes on.*

The expressions found for physical quantities on the other hand — the (II.11) or the (II.15) — all contain one component, namely population, which is increasing at a percentage rate g. Therefore, *each physical quantity increases in time at the percentage rate of growth g.*

Besides prices and quantities, there are other magnitudes in the system which are of economic interest and which are worth considering. The time-paths of two series of them in particular — the amounts of employment in each of the sectors and the production of each commodity at current prices — can be derived immediately from the movements of prices and quantities. In general, denoting respectively by E_i and V_i the sectoral employments and the productions at current prices, we may write:

$$\text{(III.12)} \qquad E_i(t) = a_{ni}(t) X_i(t),$$

$$\text{(III.13)} \qquad V_i(t) = X_i(t) P_i(t), \qquad i = 1, 2, \ldots, (n-1), k_1, k_2 \ldots, k_{n-1}.$$

Again, by substituting from (II.11), (II.14) or from (II.15), (II.16), it can be seen that, under the present assumptions, each of the E_i's and each of the V_i's (taking W as *numeraire*) increase in time at the percentage rate of growth of population (g).

There is moreover another series of magnitudes which always recur in a dynamic analysis and which we may consider immediately. I am referring to the capital-output ratios in the various sectors of the economy (which will be denoted by $\varkappa_i$, $i = 1, 2, \ldots, n-1$); and to the over-all capital-output ratio (which will be denoted by χ). Using for simplicity the formulations (II.11), (II.14) referring to a system where capital is required only for the production of consumption goods, each sectoral capital-output ratio may be expressed as (³)

$$\text{(III.14)} \qquad \varkappa_i(t) = \frac{a_{nk_i}(t)}{a_{ni}(t) + \left[\pi + \frac{1}{T}\right] a_{nk_i}(t)}, \qquad i = 1, 2, \ldots (n-1),$$

(³) In each sector, the capital output ratio is

$$\varkappa_i = \frac{K_i P_{k_i}}{X_i P_i}.$$

Remembering that K_i and X_i coincide, when full employment is kept, and using the (II.11) and (II.14), we obtain

$$\varkappa_i = \frac{a_{nk_i} W}{\left[a_{ni} + \left(\pi + \frac{1}{T}\right) a_{nk_i}\right] W} = \frac{a_{nk_i}}{a_{ni} + \left(\pi + \frac{1}{T}\right) a_{nk_i}}.$$

(continued on following page)

and the over-all capital output ratio as (4)

$$\text{(III.15)} \qquad \chi(t) = \frac{\sum a_{in}(t)\, a_{nk_i}(t)}{\sum a_{in}(t)\, a_{ni}(t) + \left(g + \pi + \frac{1}{T}\right) \sum a_{in}(t)\, a_{nk_i}(t)}$$

As can be seen, the components of (III.14) and (III.15) are nothing but coefficients. And since, under the present assumptions, all coefficients are constant through time, it follows that each sectoral capital-output ratio and also the overall capital-output ratio remain all constant as time goes on.

When the more complex formulations (II.15) and (II.16) are used, the result is

$$\chi_i = \frac{a_{nki}}{a_{ni} \frac{T - T\pi - \gamma}{T} + \left(\pi + \frac{1}{T}\right) a_{nk_i}} .$$

(4) The over-all capital output ratio is, in our notations,

$$\chi = \frac{\sum P_{k_i} K_i}{\sum P_i X_i + \sum P_{k_i} X''_{k_i}}$$

Taking the (II.11) and (II.14) and again remembering that K_i and X_i coincide when full employment is kept, we obtain

$$\chi = \frac{\sum a_{in}\, a_{nk_i}\, W\, X_n}{\sum a_{in} \left[a_{ni} + \left(\pi + \frac{1}{T}\right) a_{nk_i}\right] W\, X_n + \sum g\, a_{in}\, a_{nk_i}\, W\, X_n} .$$

$$\chi = \frac{\sum a_{in}\, a_{nk_i}}{\sum a_{in}\, a_{ni} + \left(\pi + \frac{1}{T} + g\right) \sum a_{in}\, a_{nk_i}} .$$

When the more complex formulations (II.15) and (II.16) are used, the result is

$$\chi = \frac{\sum a_{in}\, a_{nk_i} \left[1 + \gamma \frac{T}{T-\gamma} \left(\frac{1}{T} + \frac{g}{1-g}\right)\right]}{\sum a_{ni}\, a_{in} \frac{T - T\pi - \gamma}{T} + \left[\pi + \frac{1}{T} + g \frac{T}{(1-g)(T-\gamma)}\right] \sum a_{in}\, a_{nk_i}} .$$

It goes without saying that, as a straightforward corollary of the above results, all aggregate quantities — like gross and net national income, consumption, saving, investment, capital — increase in real terms, as time goes on, at the same percentage rate of growth (g) of population.

5. *Interesting features of the present case of growth*

The dynamic features of the system considered here, as they emerge from the previous sections, are of an extreme simplicity. The merits of this simplicity are entirely attributable to the assumption of a fixed technology and constant returns to scale, which confers on this case all the elegant properties it possesses. With an invariant technology and constant returns to scale, the growth of the system is entirely determined by the rate of increase of population. This growth does not affect the position of any single individual: *per-capita* income remains constant as time goes on, and economic growth simply means that the system expands all its sections in the same proportion. All products grow, with population, at the same percentage rate, while the structure of the system (its relative composition) remains constant as time goes on.

This case of growth is well known, of course, in economic theory. A clear though rudimentary picture of this case can already be found in CASSEL (5). Recently, LEONTIEF and VON NEUMANN, although in a different way (6), have based all their well-known dynamic elaborations exactly on this case.

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(5) GUSTAV CASSEL, *Theoretische Sozialökonomie,* Leipzig, 1918, pp. 27 and ff. English translation: *The Theory of Social Economy,* London, 1921, pp. 34 and ff.

(6) Both VON NEUMANN's and LEONTIEF's dynamic models will be discussed in more detail in an appendix to chapter VI.

But, most interesting of all, this is the case to which macroeconomic models of growth can be correctly applied. Since the system expands keeping its proportions constant, there is in fact no loss of generality in framing the analysis in macroeconomic terms. The usual convention that the variables must be considered as measured in terms of a composite commodity, made up of a fixed « basket of goods », is in this case perfectly legitimate and logically unobjectionable. As all our results have shown, since all single coefficients remain constant through time, the composition or structure of the system, once specified at time zero, remains the same for all time.

Chapter IV

Problems connected with technical change - Setting the bases for a general dynamic analysis

1. *Technical progress in macro-economic models*

The case of the previous chapter — despite its popularity among theoretical economists — remains a very particular case of economic growth. In practice, as soon as we look beyond a single period of time, there is another series of changes that take place besides those of population: the changes in the technical methods of production. These changes are in fact much more problematical and much more complex than those concerning population.

Technical change has been the great neglected factor in economic analysis. Only in the last fifteen years have economists begun to deal with it through the elaboration of models of economic growth. All these models with technical progress, however, have been developed *in macro-economic terms,* i.e. with the implicit assumption that one single commodity (or a composite commodity of invariable composition) is being produced in the system (1). And technical change has been in-

(1) After the path-breaking contribution of Harrod and Domar, already mentioned, the macro-economic models of growth which have been elaborated are so many as to be already difficult to count. On the whole, two main streams of thought have emerged. The first one has tried to pursue Harrod-Domar's Keynesian approach in various directions, and the second has tried to insert Harrod-Domar's ideas into the traditional neoclassical

troduced in the form of a « rate of technical progress », which has been treated exactly like, and symmetrically to, the rate of population growth ([2]).

Unfortunately, this approach has been accepted rather uncritically so far. It is my purpose to criticize and abandon it. But before doing so, I must invite the reader to take a closer look at all the implications that this approach entails. Our disaggregated formulations will turn out to be very useful in this task.

As said above, any macro-economic analysis implies that all variables considered are measured in terms of a composite commodity or « basket of goods » of fixed and invariable composition through time. Therefore, unless the macro-economic framework is given up altogether, the introduction of a rate of technical progress in such an analysis necessarily implies two further and much more specific assumptions: 1) that technical progress is going on at the same rate in all sectors of the economy; and 2) that demand for each product is expanding at the same rate.

Let us carefully consider a hypothetical case of economic growth in which these two assumptions are satisfied.

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theory. The first type of models is perhaps best represented by: J. Robinson, *The Accumulation of Capital*, London, 1956, and N. Kaldor, *A Model of Economic Growth* in « The Economic Journal », 1957. The most representative examples of the neo-classical models are perhaps: R.M. Solow, *A Contribution to the Theory of Economic Growth* in « The Quarterly Journal of Economics », February, 1956; and J. Meade, *A Neo-Classical Theory of Economic Growth*, London, 1961.

([2]) In the present and following chapters, we shall normally consider *percentage*, i.e. *relative* rates of change. However, for brevity's sake, and following what has by now become a custom, in economic literature, the words *percentage* or *relative* will normally be omitted, except in those cases where their omission may generate misunderstanding.

2. *A dynamic model with uniform technical progress and uniform expansion of demand*

After dealing with the case of population increase and constant returns to scale, the hypothetical case of economic growth taking place with uniform technical progress and uniform expansion of demand can be treated more summarily.

The assumptions are now as follows:

a) the initial conditions, as in the previous case, are such that at time zero there is full employment and full capacity utilization;

b) as time goes on, population remains constant;

c) there is the same technical progress in all sectors of the economy. This means that, as time goes on, all the technical coefficients of production decrease at a steady percentage rate ρ, i.e.

$$\text{(IV.1)} \qquad a_{nj}(t) = a_{nj}(0)\ e^{-\rho t}, \qquad j = 1, 2, \ldots (n-1), k_1, k_2, \ldots k_{n-1}.$$

d) consumers' tastes are such that the *composition* of consumption is invariant to changes in income. In other words, when income increases, each individual expands demand *of all* the commodities consumed *in the same proportion.* This means that, if *per-capita* income increases at a percentage rate ρ through time (as in equilibrium it must do), all coefficients of consumption will also increase at the rate ρ, i.e.

$$\text{(IV.2)} \qquad a_{in}(t) = a_{in}(0)\, e^{\rho t}, \qquad i = 1, 2, \ldots (n-1).$$

Under these assumptions, the two conditions for a dynamic equilibrium come out rather straightforwardly. By following exactly the same procedure as in section 2 of the previous chapter, the capital accumulation conditions emerge as

$$\text{(IV.3)} \qquad a_{k_i n}(t) = \rho a_{in}(t), \qquad i = 1, 2, \dots (n-1),$$

which are similar to the (III.6), the only difference being that now the rate of technical progress has replaced the rate of population growth. Of course in the (IV.3) all coefficients are moving, but they are moving at the same rate of change so that the relation between them remains constant through time. The effective demand condition also emerges as very similar to (III.11):

$$\text{(IV.4)} \qquad 1 - \sum \frac{1}{\alpha\beta} a_{ni}(t)\, a_{in}(t) = \left(\rho + \frac{1}{T}\right) \sum \frac{1}{\alpha\beta}\, a_{nk_i}(t)\, a_{in}(t),$$

where again the rate of technical progress has taken the place of the rate of population growth. The interesting property of this case is that all coefficients of production and all coefficients of consumption, although moving in time, are moving in an opposite direction and at exactly the same rate. As a result, *each single binary product* of coefficients under the two summations remains constant as time goes on — the movements of the components exactly cancelling each other out. This means that the contribution to national income of each single sector remains constant. As in the previous case, condition (IV.4) does not raise any problem through time. Once it is satisfied at time zero, it will remain satisfied for ever, because in all sectors productivity and demand are increasing at the same rate.

The time paths of all economic magnitudes can be found immediately by substituting (IV.1) and (IV.2) into (II.11), (II.14) and (III.12)-(III.15). As can easily be checked, if the rate of profit remains constant, the results emerge as follows:

1) physical production of each commodity increases in time at the rate ρ;

2) all commodity prices — if W is taken as the arbitrarily fixed price — decrease in time at the rate ρ. Alternatively — if the price of any commodity instead of W is taken as given — all commodity prices remain constant as time goes on, while the wage rate increases at the rate ρ. Another way of stating this result is to say that the commodity price structure (relative prices of commodities) remains constant through time, while the real wage rate increases at the rate ρ;

3) employment in each sector remains invariant as time goes on;

4) production of each commodity evaluated at current prices remains constant in time (if W is taken as the arbitrarily fixed price) or increases at the rate ρ (if any commodity price, instead of W, is taken as given);

5) all sectoral capital-output ratios, as well as the over-all capital-output ratio, remain constant.

3. *Analytical properties of the two cases of growth considered so far*

The most attractive property of the case of economic growth just examined is that it still retains the constancy of the proportions of the system in time. By the device of uniformity both in technical change and in expansion of demand, all movements of coefficients cancel out inside each sector and moreover the structure of prices remains unchanged. Thus, again, the system expands by multiplying all its sections in the same proportion so that its relative composition is invariant with respect to growth and time.

From an analytical point of view, there is a remarkable symmetrical correspondence between the two cases of growth considered so far. In the case of population growth the wage rate was constant and the system was growing at the same rate as population; in the case of uniform technical progress just examined, population remains constant and the wage rate is growing at the same rate as technical progress. The two cases may easily be combined, and the results of such a combination are so straightforward that it is of no use to spell them out in detail here. We may however explicitly state at least the conditions of equilibrium, which emerge as follows:

$$\text{(IV.5)} \qquad a_{k_i n}(t) = (g+\rho) a_{in}(t), \qquad i = 1, 2, \ldots, (n-1),$$

and

$$\text{(IV.6)} \qquad 1 - \sum \frac{1}{\alpha\beta} a_{ni}(t)\, a_{in}(t) - \sum \frac{1}{\alpha\beta T} a_{nk_i}(t)\, a_{in}(t) =$$
$$= (g+\rho) \sum \frac{1}{\alpha\beta} a_{nk_i}(t)\, a_{in}(t)\,.$$

The economic meaning is evident. Each single sector of the system and the system as a whole expand at a rate which is the sum of the rate of population growth and of the rate of technical progress, a sum which is widely known in economic literature by HARROD's term of *natural rate of growth*. This natural rate appears explicitly both in (IV.5) and in (IV.6), the latter now giving a complete analytical break-down of HARROD's equations. (The aggregate net saving ratio is required to be equal to the natural rate of growth multiplied by the over-all capital-output ratio) (3).

At this point, however, after admiring the symmetry and the analytical beauty of the two cases of growth considered so far, we must also draw from the results we have obtained at least two logical and stringent conclusions.

(3) See the proof in footnote (1) of Chapter III.

The first one refers to methods of analysis. If the assumptions embodied in the two cases of economic growth considered above were acceptable, then there would not really be much gain of insight into the working of an economic system by using a disaggregated model. The merit of such a model would only be to show the structure of the system at a given point of time. But since this structure remains the same for ever, the dynamics of the system is always uniform, which means that no extra information can be obtained by disaggregating. In other words, a macro-economic formulation would be by itself sufficient and satisfactory.

The second conclusion is of a much more practical relevance. If the assumptions embodied in the two cases of economic growth considered above were to correspond, even roughly, to what in the long run is happening in the real world, then any preoccupation about problems of economic growth would be entirely unjustified. The model considered above amounts in fact to saying that economic growth *as such* does not present any problem at all. The only thing that in any economic system is to be done is the setting up of that particular structure which is most desired — the only constraint being that it must satisfy relations (IV.5) and (IV.6). This is a *once-for-all problem*. Once this structure has been set up, no problem exists any more. Thereafter, the system will expand for ever, keeping proportions constant.

Attractive though the first conclusions may be, the second one is so much in a striking contrast with everyday experience and with the economic policies of all Governments, that it should immediately lead one to infer that there must be something wrong somewhere. And it is my contention that this something wrong is to be found exactly in the hypothetical case of economic growth considered in the previous section.

To substantiate this assertion will require all the rest of the present chapter.

4. *The production aspect of technical change*

It will now be useful to consider carefully the meaning and implications of technical change. When the technical coefficients of production change as time goes on, there are two distinctly different series of effects which are called into being and which may respectively be connected with production and demand. On the production side, technical change means a variation in the technical conditions and therefore a change in the physical quantities of goods which may be produced out of a given amount of original factors of production. On the demand side, it means a change in remuneration of the factors of production, and therefore a change in the amount of *per-capita* real income at the disposal of consumers in general.

Let us take the production aspect first. Here the causes of change may be manifold. For a long time economists have been impressed by those changes which are connected with the exhaustion of natural resources. To MALTHUS and RICARDO, for example, at the beginning of last century, it appeared as a matter of logical necessity that the continuation and expansion of the process of production, on natural resources which are given, should inevitably lead to decreasing returns, i.e. to an increasing trend in the technical coefficients of production. But the economic history of the industrial countries has by now consistently and persistently brought to the fore another, more important and widespread process of change, continuously at work in any modern society: technical progress.

Technical progress is a very complex movement. In the sense in which it is relevant for economic analysis, it includes not only, and not so much, the great scientific discoveries, which by their own nature come about in a discontinuous and sometimes accidental way, as their practical application on an industrial scale, which takes place through a much longer and continuous process. Moreover, it includes all the innumerable series of expedients and devices, small if considered individually, but of great relevance if jointly gauged, which are the daily upshot of experience, experiments, research and of re-thinking of the productive organization. This is indeed a very complicated process, emerging from the *learning* activity of men and the application of this learning activity to production. By its nature this process is, therefore, a slow but persistent one. It consists of long and repeated attempts not only to reorganize the old methods of production, to utilize more efficiently the new materials, and to improve the quality of the products, but also to invent and apply new methods of production, produce new products, find new resources, and discover new sources of energy.

It would certainly be out of place to develop a *theory* of technical progress here. If such a theory should ever be developed, it would pertain to a much wider field than economics, because it could not avoid some definite conceptions about the aims and ends of human society. Therefore, as far as the present work is concerned, the usual procedure will be followed of taking technology as *given* from outside economic analysis. However, what will be taken as given is not a fixed technology but the *movements* of the technical coefficients in general through time.

In this way, the concept of technical change which is adopted in the following pages will not be restricted by any particular assumptions ([4]). But in order to reach practically

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([4]) It will cover the case of technical progress of any type and also the case of increasing or decreasing returns to scale. On the other hand, the case on which neo-classical economists focussed all their attention (a change of production methods owing to changes in the rate of profit) will automatically not arise because in the present model the rate of profit remains constant as time goes on.

relevant conclusions, the analysis will be carried out with reference to those movements that empirical findings have by now shown to be everywhere the most typical ones in a modern society. These movement may be briefly stated in three propositions:

a) in the long run, the effects of technical progress are, on a (weighted) average, by far more important and more widespread than the effects of decreasing returns to scale (5). This means that, as time goes on, the coefficients of production decrease (i.e. productivity increases) in most sectors, although in a few sectors the coefficients might increase (i.e. productivity decrease);

b) as a net effect of decreasing returns and of technical progress, each production coefficient is slowly but persistently moving through time. However, each coefficient is moving at a different speed. In other words, there is a wide dispersion amongst the rates of change of productivity referring to the different branches of the economy (6);

c) technical progress consists not only of increases in productivity but also of continuous additions of new sectors producing new and better goods for the economic system.

5. *The demand aspect of technical change*

Let us now consider the effects of technical change on demand. If on the whole technical change is in the direction of a persistently increasing trend of productivity, it means a

(5) The simplest empirical confirmation of this proposition is that in all industrial countries, *per-capita* income is enormously higher today than it was when they began to industrialize.

(6) Cf. for example the interesting study by F. L. Hirt, *a new Look at Productivity Growth Rates*, in « Survey of Current Business », 1957.

higher and higher amount of wages and profits, or, more generally, an increasing trend in *per-capita* incomes at the disposal of consumers. It follows that, in each period of time, technical progress compels the members of the community to make new decisions; they must decide on which commodities they are going to spend the increments of their incomes. It is here that their preferences as consumers come to play a central rôle.

Consumers' preferences ultimately depend on the character of human nature, which represents, in the same way as the technical conditions of production do, a fundamental datum for any meaningful economic investigation. No commodity, whatever ingenious technique it may require, can be sucessfully produced if its utility for the consumers is not sufficient to justify its cost: it would remain unsold. The relevance itself of technical progress depends on demand; an increase in productivity, however large it may be, loses much or even all of its meaning, if it takes place in the productive process of a commodity for which demand is small or negligible. This means that any investigation into technical progress must necessarily imply some hypotheses (and if not explicitly, it necessarily does so implicitly) on the character of the evolution of demand as income increases. Not to make such hypotheses and to pretend to discuss technical progress without considering the evolution of demand would make it impossible to evaluate the very relevance of technical progress and would render the investigation itself meaningless. Increases in productivity and increases in real income are two facets of the same phenomenon. Since the first implies the second, and the composition of the second determines the relevance of the first, the one cannot be considered if the other is ignored.

Unfortunately, the economic theory which has so far been developed is hardly able to give us any help on this problem. The consumers' demand theory that we know today is a highly sophisticated logical framework, built on static premises. It relies on well-known and consistent preferences defined *at a given level of per-capita income*. Such a theory is indeed useful in showing the consequences of price changes, at a given level of income, but has nothing to offer us to explain changes following each successive *increment* of income.

The regrettable consequence has been, that when economists — by introducing technical progress in their models — have been *compelled* to make definite hypotheses about the expansion of demand, in the absence of any guiding principle, have made those assumptions that best suited the mathematical properties of their models. As we have seen, they have invariably postulated, either explicitly or implicitly, a uniform and proportional expansion of *per-capita* demand (7).

Now, if there is something that we do positively know about expansion of *per-capita* demand when income increases, it is that *per-capita* demand for each commodity *does not expand proportionally*. All the empirical investigators who, in the last hundred years, have looked into this matter have invariably and without exception confirmed this tendency.

As is well known, the first empirical generalizations on the evolving pattern of demand, in response to increases in income, come from an old discovery in economics which goes back to Ernst Engel (8) in the 1850's. Engel, after studying the conditions of consumption of workers in the kingdom of Saxony, stated what has since become known as *Engel's law*. The law says that the proportion of income spent on food declines as income increases. A more general formulation of this

(7) The effect of this is that all models of economic growth now-a-days share a defect which was characteristic of Classical analysis. They have concentrated their emphasis exclusively on the production side of the economic process and have entirely forgotten the other half of economic reality, related to demand.

(8) Ernst Engel, *Die Productions- und Consumptionsverhältnisse des Königreichs Sachsen*, in « Zeitschrift der Statistischen Bureaux des Königlich Sächsischen Ministerium des Inneren », Nos. 8 and 9, Nov. 22, 1857; republished in « Bulletin de l'Institut International de Statistique », IX (1895).

empirical law, stating that the proportion of income spent *on any type of goods* changes as *per-capita* incomes increase has been confirmed ever since, and evinced by all the econometricians who have been concerned with empirical work on demand (9).

These results are in fact no surprise. It should not take long to realise that what they reveal is a basic tendency inherent in the human nature of the consumers. We shall go into this matter in the next section. Meanwhile we may safely draw the conclusion that all the models of economic growth so far developed have adopted a set of hypotheses which are incompatible with one of the most fundamental empirical laws of economics. Since increases in *per-capita* income necessarily imply non-proportional expansion of demand, and since technical progress means increases in *per-capita* incomes, then the introduction of technical progress in any dynamic model necessarily implies a non-proportional expansion of demand. The assumptions adopted in all macro-economic models of growth are therefore unacceptable.

6. *The evolution of demand in time*

The practical importance of ENGEL's law has always been recognized by all those who, on any occasion, have been engaged in empirical work on demand. Yet, in spite of the fact that it was discovered more than a century ago, very little work has been done to try to take advantage of the information it gives and to incorporate it into the theory of consumer's behaviour. (Bits of piece-meal theory can be found only in the works of the econometricians who — faced with facts — have always been compelled to make additions and to adapt an in-

(9) See, for example: R.G.D. ALLEN and A.L. BOWLEY, *Family Expenditure*, London, 1935; and also: H.S. HOUTHAKKER, *An International Comparison of Household Expenditure Patterns, Commemorating the Centenary of Engel's Law*, in « Econometrica », 1957.

sufficient theory to their particular purposes). As a result, ENGEL's discovery has remained until our own day at the state of an empirical law, almost entirely isolated from the body of demand theory.

It is not my purpose, of course, to engage in a theoretical investigation on demand here. However, since the tendency that ENGEL discovered plays a central rôle in the subsequent analysis, I shall discuss at least two theoretical aspects of it, and make to current demand theory (otherwise insufficient for our purposes) a few essential additions.

The first basic point I should like to make refers to the nature of human preferences. This point applies even if we supposed that individual preferences can be represented by perfectly known utility functions. It would be misleading to ignore, it seems to me, what we do know on the subject, namely that the utility that any commodity can give depends on the previous consumption of other commodities. For, the activity of consumption is a process in which there is a very definite order of succesive steps to be taken. For example, the decision to buy a motor-car presupposes that the consumer has already bought — and is permanently in the condition to buy — an adequate quantity of food, clothes, dwelling space, etc. The motor-car would not have for him the same utility if he were not well-fed, well-clad, etc. ([10]). In other words, the utility itself of the motor car depends on the type and quantity of commodities which have been consumed already. To talk of these problems evidently means talking of the shape of consumers' preferences considered as a whole (i.e. of the whole utility function); it means talking of *absolute levels* of utility (and not only of *marginal* utilities). Let me point out that there are in particular some basic human needs (like eating and breathing) for which the commodities that are necessary may be said to have an infinite level of utility: without them men would die!

([10]) This interdependence was first pointed out to me by Mr. N. KALDOR.

Yet, once these needs are satisfied the marginal utility of successive increments of those commodities may fall dramatically and very quickly even become negative. Obviously, before this happens, demand will shift to other commodities. But the property that those basic needs saturate rapidly in no way alters the fact that they must be satisfied first of all.

Now, current demand theory (especially — I must say — after its recent elegant refinements) has focussed our attention *exclusively* on what happens *at the margin* — on marginal substitution among commodities if a price change, or on equalisation of *marginal* utilities. Such an approach may have a justification in a static world, where everything which may happen cannot but happen at the margin (which always remains the same). But in a world where *per-capita* incomes are moving, there is very little help we can get from marginal utilities, unless they are specified over the whole range of the utility functions, i.e. unless we pass from them to absolute levels of utilities. It is the absolute level of utility of each commodity that will tell as which of the various commodities comes next into the range of consumer's preferences, even if the corresponding want will then rapidly saturate.

To conclude, we may say that, owing to a fundamental property of human nature, there exists a very definite order of priority in consumers' wants. The less basic a want is, the higher will be the number of wants that must be saturated before it can be afforded consideration. This order of priority is especially strong at low levels of income, where satisfaction of some wants is a *conditio sine qua non,* even to the appearance of all the others. But the order persists at a higher level as well, where the process of decision becomes more complicated only because the order itself may no longer be so obvious, and needs first to be discovered. This takes me to my second point.

The second addition to demand theory that I should like to make refers to the nature of the behaviour of human beings.

Here, once more, traditional theory has taken up only a limited case to study. It has always assumed that individuals know their preferences perfectly and behave rationally. There is of course a justification for such an approach in a static environment, but no longer in a dynamic world. We do positively know that human beings are not omniscient, and that the way in which they come to know new situations is through experience. They may know reasonably well the problems faced already in the past; for they have had the opportunity of trying and experimenting with different solutions. But if a new situation arises, they have first to learn how to deal with it, and the decisions they make the first time may not be the best ones — they are tentative decisions in order to learn. Now, if real *per-capita* income is continually rising, each consumer enjoys an extra amount of income to spend in each successive period, which indeed puts him in a new situation. Especially when incomes become high, to pretend the consumer makes the best decision — according to his preferences — about the extra income he has just obtained is unreasonable. He *does not know his preferences* at that high level of income, because he never experienced them before — he has *to learn them*. This is not all. As time goes on, the quality of old goods may change, and the price structure may also change; while old needs may be satisfied with better (superior) goods. This means that the consumers' learning activity is a process required over the whole range of his preferences. As a conclusion, I should like to propose here to enlarge our views of consumers' behaviour and to complete the traditional postulate « the consumer is a rational being » with the more general one « the consumer is a learning being ». The latter is more general because it can be regarded as including the former as a particular case: the case of a stationary economic system. (In such a system technology and income have, by hypothesis, always been constant through time, which means that all learning activity has been completed already in the past. Since consumers had infinite opportunities of trial and error, they must now know their preferences perfectly. The second postulate thereby reduces to the first).

This is enough for our purposes. To synthesize these results for the following analysis, we may simply state them in three propositions:

a) at each level of *per-capita* income, the proportion of income spent on any commodity is generally very different from one commodity to another;

b) as *per-capita* real income increases, each increment of demand tends to concentrate on a particular group of commodities. This group of commodities changes from one level of income to another (it may be mainly food at very low levels of income, clothes and again food at slightly higher levels of income, houses, durables and services of various kinds at further higher levels, etc.). In other words, as income increases, the tendency of the consumers is not to increase proportionally the consumption of already-bought commodities, but rather to buy new goods and services or also to satisfy old needs with different (better) goods;

c) there is no commodity for which any individual's consumption can be increased indefinitely. An upper saturation level exists for all types of goods and services, although at different levels of income: it may be reached sharply — in the case of goods satisfying physiological needs — or only through a slow and long process as income increases — in the case of services yielding very sophisticated types of satisfaction — but its attainment is eventually inevitable. Moreover, demand for some particular goods (inferior goods) may in fact decrease, after reaching saturation, if real income persistently goes on growing.

If we want to represent these results graphically, by plotting the expenditure for each commodity as a function of real income, we should obtain families of curves of the following shapes:

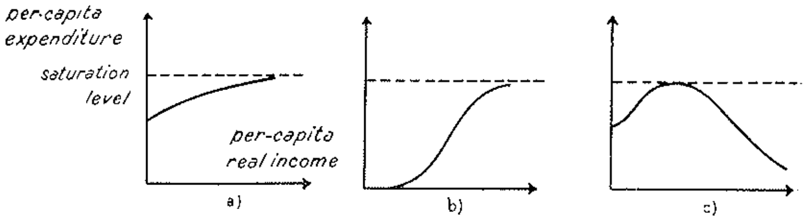

FIG. 1

Curves of type (*a*) are likely to fit the cases of goods which are absolutely necessary for physiological reasons (e.g. food), and curves of type (*b*) are likely to fit almost all other cases. Type (*c*) finally represents the typical behaviour of inferior goods ([11]).

Of course, the relation between expenditure on each single commodity and real income, represented in fig. 1, is limited by the two-dimensional character of the diagrams. The actual time-path of each single expenditure will also depend on the variation of the structure of prices. It is important to realize, however, that the *shapes* of the relations represented in fig. 1 will remain unaffected.

7. *The criterion for the choice of the hypotheses*

Before going on to considering the general dynamic model to which the foregoing discussion has gradually paved the way,

([11]) J. AITCHINSON and J.A.C. BROWN (in *The Log-normal Distribution*, Cambridge, 1957, chap. 12) suggest a function based on the log-normal distribution, as a general function capable of fitting almost all the cases of ENGEL curves. The shape of the function is of the type (*b*) represented in fig. 1.

it may be useful to face explictly a spontaneous question which might be raised at this point. Why should the subject of economic growth be approached in this way? The theoretical scheme of the previous pages is, after all, full of simplifications and rough approximations. Now, why should such simplifications be made in preference to others? And in particular, why should such simplifications be made in preference to those that traditional economics has adopted so far?

This is a relevant question to ask because, as pointed out already, any model does entail a simplification of reality and it is of paramount importance to be clear about the criterion according to which assumptions are made. One is bound to make immediately some simplifications in the choice of the *variables* and of the *constants* of any economic enquiry. For, in economics, there are — rigorously speaking — no constants. All quantities one may be dealing with are, in fact, variable. But since some quantities are more variable than others, it is commonly said that one should take as constants those quantities which vary the least. This clearly is a reasonable criterion to follow, provided that it is applied consistently. But there has been difficulties, connected with the development of economic thought.

In the last hundred years, economists have just happened to be interested in the static characteristics of an economic system, or at most in its short-term behaviour. And since in the short run hardly any quantity can vary substantially, they have tended to make the distinction between variables and constants coincide with the distinction between unknowns and data. Those economic magnitudes that were to be explained (unknowns) have also been taken as variables, and those magnitudes that were to be accepted as given (data) have also been taken as constants.

Obviously, such an association did not matter very much as long as it was restricted to a short-run analysis. But when, recently, the attention of economists has shifted to dynamic problems, some difficulties have began to arise. For, some data — i.e., factors to be accepted as given from outside economic analysis — are in fact much more variable in time than some of the economic magnitudes taken as variables. In such circumstances, the traditional association has made the choice between variables and constants slowly undergo a serious distortion. Some magnitudes have been kept as variables although their variability is not important, and other magnitudes, whose variability is an essential feature of the long-run growth pattern, have been left out of the analysis altogether.

It is precisely to avoid this distortion that the theoretical framework developed in the present work has been laid down, straight from the beginning, with reference to the requirements of a long-run dynamic analysis. The criterion for the choice of the hypotheses is a consequence of this approach. I have considered as typical *variables* those quantities which — independently of whether they are to be explained or not — undergo changes of an irreversible character as time goes on, by incessantly moving on in the same direction. Although the changes might be quite negligible within a single short period of time, each period marks for these quantities a step forward, in a slow but cumulative movement. On the other hand, I have assumed as *constants* those quantities which, in the long run, do not present any tendency (or for which there is no reason to expect any tendency) to move in any direction. These quantities, of course, may change quite a lot from one period to another, but the point is that their changes are temporary and reversible. Even if they do go for some time in a particular direction, they cannot go on indefinitely, and they are bound to come back to where they started.

The difference between these two types of quantities can be immediately perceived if we consider a long period of time. Compare, for example, the American economy in 1860 and in 1960. It is quite possible and easy to claim that, within this period, magnitudes like the average time-life of the equipment or the rate of profit have undergone considerable variations, as shown, for example, by a coefficient of dispersion. But the important point is that such a dispersion, if it has taken place, has done so around a roughly constant trend. The result is that, after a century of vicissitudes, these magnitudes are in 1960 practically at the same level as they were in 1860. The case of population, *per-capita* income, or composition of demand, is radically different. These quantities may have changed very little from one year to another but they have always changed in the same direction. The cumulative result, after a century, is enormous. Population has increased five times, *per-capita* income has also quintupled, and total consumption is mostly composed in 1960 of goods and services that in 1860 did not even exist: in other words, the trends of these variables have been irreversibly and persistently increasing, and are going to persistently increase in the future.

To conclude, the distinction between variables and constants in the preseut analysis has been based on variability through time. Therefore, it does not coincide, and must not be confused with, the distinction between unknowus and data, namely between quantities which are intended to be explained and quantities which are accepted as given from outside economic analysis. Accordiugly, there are magnitudes — such as population and technical progress — which are here taken as given from outside economics and nevertheless are essential variables. And there are other magnitudes, like the rate of profit, which are to be explained by economic investigation, and which nevertheless have been taken as constant over time.

CHAPTER V

A GENERAL MULTI-SECTOR DYNAMIC MODEL

1. *The model*

We are by now in a position to expound a very general multi-sector dynamic model. After the foregoing analysis, the exposition need not be long and we may proceed in a very schematic way. The hypotheses which are made will be listed here under two headings, referring to initial conditions and to movements through time.

A. *Initial conditions*. — At the time when our analysis begins, which we may denote as time zero, the system is *in equilibrium*, i.e. there is full employment and full utilization of existing productive capacity. The system is defined by:

a) a series of $(n-1)$ stocks of capital:

$$K_1(0),\ K_2(0),\ \ldots\ K_{n-1}(0)\ ;$$

which are the result of past productive activity. For simplicity, the assumption will be kept that capital goods require no capital to be produced. Since the system is in equilibrium, the stocks of productive capacity are exactly of the size required by demand;

b) a population $X_n(0)$, which is taken as an exogenous magnitude. The flow of labour services which this population can provide in each unit of time is equal to $X_n(t)$ divided by two coefficients — $\alpha(t)$ and $\beta(t)$ — standing for the proportion of active to total population and the proportion of actual working time to the total time composing the time unit respectively. The subscript t has been added to both coefficients because they may be undergoing a long-run trend;

c) a series of $2(n-1)$ technical coefficients:

$$a_{n1}(0), \dots a_{n,n-1}(0), \dots a_{nk_1}(0), \dots a_{nk_{n-1}}(0) \ ;$$

expressing the inputs of labour required in the unit of time — in combination with the appropriate stock of capital — to produce one physical unit of final commodity. There is also a series of $(n-1)$ technical coefficients $T_1, \dots, T_{n-1}$, which can roughly be interpreted as expressing the average life-time of capital goods in each sector where capital goods are required;

d) a series of $(n-1)$ demand coefficients [1]:

$$a_{1n}(0), \dots a_{n-1,n}(0),$$

expressing *per-capita* demand for each consumption good in the unit of time. There is moreover another series of $(n-1)$ demand coefficients for new investments:

$$a_{k_1 n}(0), \dots a_{k_{n-1} n}(0).$$

(1) It may perhaps be useful to point out explicitly that although the technical coefficients and the *per-capita* demand coefficients occupy a symmetrical place in the system, they are not of the same nature. Technical coefficients represent *sectoral* concepts. Each of them is given by the state of technology in each particular sector. *Per-capita* demand coefficients represent *macro-economic* concepts. Each of them is an average taken all over the system.

These coefficients, however, cannot be taken as exogenously given. How they are determined will be discussed in section 4. At time zero, since the system is in equilibrium, all demand coefficients considered together are such as to require the full utilization both of the labour force and of the existing productive capacity.

B. *Movements through time.* — As time goes on, the following movements take place:

a) population increases at a steady percentage rate g, so that

$$X_n(t) = X_n(0)\ e^{gt}\ ; \tag{V.1}$$

b) productivity changes at a particular percentage rate of change in each sector. It will be assumed that these rates of change (ρ_j) are different from one sector to another but that they are steady through time in each sector. This means that

$$\text{(V.2)}\qquad \begin{aligned} a_{nj}(t) &= a_{nj}(0)\ e^{-\rho_j t}\ , \\ a_{nk_j}(t) &= a_{nk_j}(0)\ e^{-\rho_{k_j} t}\ , \qquad j = 1,\ 2,\ \ldots\ (n-1). \end{aligned}$$

Most ρ_j's and ρ_{k_j}'s ($j = 1,\ 2,\ \ldots\ n-1$) are positive, but a few of them (referring to those sectors where the exhaustion of natural resources is particularly heavy) might be negative:

c) per-capita demand changes at a particular percentage rate of change for each commodity. We shall denote these rates of changes as r_i, ($i = 1,\ 2,\ \ldots\ n-1$). The r_i's *are not constant over time*; they change as a result of a very complex process, as has been explained in the previous chapter. The r_i's as such are not exogenous magnitudes, in our analysis. What has been assumed as an exogenous datum is the set of consumers' preferences. Thus, as time goes on, the shape of these consumers' preferences and the movements of technical coefficients postulated by (V.2) determine the time movements of the r_i's. We may write:

$$(V.3) \qquad r_i(t) = f_i \left\{ a_{n1}, \dots a_{n,n-1}\, a_{nk_1}, \dots a_{nk_{n-1}}; \right.$$

$$\left. \frac{d}{dt} [a_{n1}, \dots a_{n,n-1}, a_{nk_1}, \dots a_{nk'_{n-1}}] \right\},$$

$$i = 1, 2, \dots (n-1);$$

where the f_i's depend on the shape of the consumers' preferences, which are defined in such a way as to satisfy propositions *a*), *b*), and *c*) of chapter IV section 6, graphically represented in fig. 1. The technical coefficients that appear in (V.3) influence the r_i's through the medium of two channels: the level and the rate of change of real *per-capita* incomes, and the variation of the structure of prices.

Here, for purposes of simplicity and symmetry, the assumption will be made that the movements of *per-capita* demand may approximately be broken down (if we represent them on a logarithmic diagram) into stretches of straight lines. In other words, we assume that time can be divided into finite stretches of length s (larger than the unit of time we are adopting) within which, for each commodity i, the percentage rate of change of demand r_i remains constant. Then passing from one stretch of time of length s to the next, r_i changes, remaining then constant again for another stretch of time s. And so on.

Therefore, by defining now a new variable θ as

$$\theta = t - \mu s ,$$

where μ is the greatest integer that multiplied by s and subtracted from t leaves a positive remainder (θ), the movements through time of the demand coefficients may be written as (²)

$$(V.4) \qquad a_{in}(t) = a_{in}(t-\theta)\ e^{r_i\theta}\,, \quad i = 1, 2, \ldots (n-1),$$

where each r_i is an f_i function of the technical coefficients and now also of $(t-\theta)$. In order not to complicate the notation excessively, this functional dependence is not explicitly written in (V.4), nor will it be written hereafter, but it must always be taken as understood.

Of course, the same notation in terms of θ can be used also in formulations (V.1) and (V.2), and this will be done, in the following analysis, any time it is required by reasons of symmetry.

2. *A few restrictions*

First of all, it may be useful to impose a few restrictions on the coefficients of our system. These restrictions do not imply anything new and in fact have always been implicit in our previous analysis. We only make them explicit now for the sake of rigour. The reason for these restrictions is that our mathematical terms are very general and their generality needs to be limited to the range in which it has economic sense.

(²) As explained in footnote (1), demand coefficients represent average *per-capita* consumption. No discussion is carried out here about possible complications arising from a changing distribution of income among individuals or from a changing composition of the population as regards sex and age. Evidently, the simplest way of interpreting our analysis is to suppose that both these features remain invariant in time. However, even if they should change, their changes can be neither quick nor big. In any case, they would not affect the conclusions of the following analysis because their effect would simply be to anticipate or postpone turning points, without altering the nature of the trends through time.

To begin with, all technical coefficients, whether referring to the production of consumption goods or of investment goods, and all demand coefficients of *consumption* goods can never be negative. In mathematical terms:

$$
\text{(V.5)} \qquad
\begin{aligned}
&a_{nj}(t) \geq 0 \qquad && j = 1, 2, \ldots (n-1),\\
&a_{nk_j}(t) \geq 0 \qquad && j = 1, 2, \ldots (n-1),\\
&a_{in}(t) \geq 0 \qquad && i = 1, 2, \ldots (n-1),
\end{aligned}
$$

which have a quite straightforward meaning. For, a negative amount of labour or a negative amount of consumption makes no economic sense.

On the other hand, the coefficients of demand for net investment *can* become negative, although only to a certain extent, i.e. up to that point at which gross investment is still non-negative. The restriction applies with absolute certainty to *total* gross investment. It need not necessarily apply to gross investment in each single sector, if capital is flexible enough to allow some transfers of capital from one sector to another when needed. In other words, taking first the most restrictive of all cases,

$$
\text{(V.6)} \qquad \frac{1}{T_i}\, a_{in}(t) + a_{k_{i}n}(t) \geqslant 0, \qquad i = 1, 2, \ldots (n-1).
$$

Should a situation arise in which the equilibrium conditions (which will be discussed in a moment) would require some of inequalities (V.6) to be reversed, then some more information about existing capital stocks is needed in order to tell what will happen. If, in those sectors where inequalities (V.6) are required to be reversed, capital is very specialized and cannot be transferred anywhere else, then idle capacity (i.e. a discontinuity in the model) will appear. If, on the other hand, capital is perfectly adaptable and can be transferred to other processes of production, then even negative gross investments can take place in those sectors.

In no case, however, as said above, can *total* gross investment become negative. In other words, the inequality

$$\text{(V.7)} \qquad \sum \left[a_{k_{i}n}(t) + \frac{1}{T_i} a_{in}(t) \right] \geqslant 0$$

must always and in any case be satisfied.

3. *The flows of the system*

Within each single period of time (finite or infinitesimal as it may be) there are flows of commodities from the production processes to the final sector and flows of labour services from the final sector to the production processes. These flows have been examined already in the short-run inquiry of chapter II, and need not be further discussed here. Within each single period of time, the structure of the system is represented by systems (II.9) and (II.13), and we may consider these two systems as rewritten here. The only difference is that we must now add a time subscript to each single coefficient and to each single variable, so that we obtain a pair of those systems for each unit of time we are considering. And since a very specific set of hypotheses has been made about how the coefficients of production and of demand change as time goes on, we shall now at last be able to look into the whole dynamics of the system. In other words, after finding the solutions of the equations in each particular period of time, we shall be in a position to look at the *movements* of these solutions through time.

As explained in chapter II, the rate of profit (π) which appears in system (II.13) will be taken as a constant. This does not mean that the rate of profit is an exogenous magnitude. It only means that it does not depend on the structure of the economic system, as represented by (II.9) and (II.13), and must be explained with a separate economic theory. Now, since I have had the opportunity of dealing with such a theory already in an independent publication of mine (*Rate of Profit and Income Distribution in relation to the Rate of Economic Growth* in « The Review of Economic Studies », October 1962), I shall here simply refer the reader to that publication (3). The results of that analysis which are relevant for our present purposes are that in an economic system where the over-all rate of capital accumulation remains roughly steady over time, the rate of profit is bound to remain steady as well.

A further point is that the assumptions of section 1 do not yet provide us with the dynamics of all coefficients that appear in systems (II.9) and (II.13). The movements through time of one series of coefficients — the demand coefficients for new investments — have not been specified. This series of structural coefficients is the only one that affects the stocks of the system (i.e. productive capacity in each sector). It cannot therefore be taken as fixed from outside the system. These coefficients must be such as to be compatible with the process of growth and will come out themselves to be determined as a part of the equilibrium conditions.

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(3) To those readers who still find it difficult to follow the logic of the long-run-equilibrium growth models I would suggest a device. We have normally been used to think in terms of a free market economy and then to extend the results to the case of a centrally planned economy. Here, it turns out ot be much more helpful to reverse the procedure and to think first in terms of a centrally planned economy. For, in this case, the relationship (an equilibrium relationship) between the long-run rate of profit and the natural rate of growth emerges immediately. The corresponding relationship for a free market economy will then appear much easier to grasp.

4. *The conditions for a dynamic equilibrium*

According to our assumptions, when our analysis begins (time zero), the economic system we are considering is in equilibrium — there is full employment and full utilization of productive capacity. We know already that, if this equilibrium situation is to be kept, two types of conditions must be satisfied.

First of all, since both population and technology are changing, the system must continually enlarge its productive capacity so as to keep up both with the increasing demand and with the increasing labour force. This means that, in each sector, a very definite relation must be satisfied between new investment and the rate of change of the corresponding final demand for consumption goods. The problem has been discussed already in section 2 of chapter III. By following here exactly the same procedure, the mentioned relations emerge as

$$\text{(V.8)} \qquad a_{k_i n}(t) = (g + r_i)\ a_{in}(t), \qquad i = 1, 2, \ldots (n-1),$$

which represent the *capital accumulation conditions* for keeping full employment over time.

After explicitly inserting the dynamic movements of demand and of population that have been postulated, the (V.8) may also be written as

$$\text{(V.9)} \qquad a_{k_i n}(t) = (g + r_i)\ a_{in}(t - \theta)\ e^{(g+r_i)\theta}\ ,$$

or, if we refer them to total net investment in each sector (instead of referring them to *per-capita* investment), they may also be written as

$$\text{(V.10)} \qquad X''_{k_i n}(t) = (g + r_i)\ X_n\ (t - \theta)\ a_{in}(t - \theta)\ e^{(g+r_i)\theta}\ .$$

These relations define equilibrium investment in each single sector, and therefore their sum defines total equilibrium investments in the system as a whole. As can be clearly seen, equilibrium investments, in physical terms, are exclusively determined by the expansion of demand.

Yet, as we know, the actual undertaking of all these investments only provides the required productive capacity, i.e. *potential* full employment. For the system to reach full employment, a further macro-economic condition about total effective demand must also be satisfied.

In mathematical terms, all this is expressed by the fact that conditions (V.9) only come to complete the two systems of equations (II.9) and (II.13), by defining at any point of time the series of coefficients (demand coefficients for new investments) which was still missing. But the two systems of equations, thus completed, are of the linear and homogeneous type, and their coefficient matrix must be singular if they are to yield non-trivial solutions. This condition of singularity, which was expressed by (II.20) with reference to a particular period of time, must now be kept satisfied over time. After substituting (V.9) into (II.20) and after explicitly expressing the movements of the coefficients, we obtain

$$\text{(V.11)} \qquad 1 - \frac{1}{\alpha\beta} \sum a_{ni}(t\text{-}\theta)\, a_{in}(t\text{-}\theta)\, e^{(r_i - \varrho_i)\theta} = \frac{1}{\alpha\beta} \sum \left(\frac{1}{T_i} + g + r_i \right) a_{nk_i}(t\text{-}\theta)\, a_{in}(t\text{-}\theta)\, e^{(r_i - \varrho_{k_i})\theta} \quad ,$$

which finally represents, under our present hypotheses, *the effective demand condition* for keeping full employment over time.

This expression, as pointed out earlier, refers to the economic system as a whole. The effect of substituting into it expressions (V.9) is that now (V.11), besides stating the general condition that total expenditure in the system as a whole must be equal to potential total income, also specifies a very definite division of this total expenditure between new investments, replacements and total consumption. Replacements are determined by the rate of wear and tear, and new investments by the rate of change of demand for consumption goods. The sum of these three types of demand — as a proportion of total potential gross income — is required to be equal to unity.

The right hand side of (V.11) represents gross equilibrium investment expressed as a proportion of gross potential income and the left hand side the proportion of gross income which is not spent on consumption goods. Let me point out explicitly (since a misunderstanding is possible) that the left hand side of (V.11) may, therefore, be regarded as expressing the saving ratio, but nothing more than that. It must not be interpreted, for example, as a *propensity* to save. No behavioural relation about savings has been introduced in the present analysis. We have simply been looking for the conditions that must be satisfied to keep the system in equilibrium, independently of how individuals behave and of how institutions induce them to behave. And our results have been that — whatever individual behaviour may be — the equilibrium conditions are two. First of all, enough productive capacity must be provided, in each sector, to keep up with the expanding demand — condition (V.10). And secondly, given total equilibrium investment required by all the (V.10), the system as a whole must spend on consumption goods the whole remaining part of gross potential income — condition (V.11).

Total equilibrium investment — i.e. the right hand side of (V.11) — may of course be given the macro-economic interpretation we have already discussed. We may say that, in macro-economic terms, (V.11) expresses HARROD's equation [4].

---

[4] See footnote (1) in chapter III.

However, we may notice that we can no longer stop our analysis at this point in the present model. The fulfilment of macro-economic relation (V.11) is now no longer independent of time and of the structural dynamics which is going on behind all coefficients.

5. *Relevance of a disaggregated formulation*

We may now begin to assess the relevance of the disaggregated type of analysis which has been developed in the previous pages. Condition (V.11) — as said above — states an important macro-economic conclusion, which is the same that has emerged from all macro-dynamic models. But this conclusion now emerges as expressing only what appears on the surface of the whole problem of a dynamic equilibrium. Relation (V.11) also implies that — just in order that it may be satisfied as an overall condition — a very complex process of structural change *must* go on behind all the macro-economic magnitudes. This structural process is in fact what technical progress means in a modern society.

We may note that to have shown the existence, and now to make possible the analysis, of this process of structural dynamics is one of the main innovations of the model developed in this chapter. The two simpler models, discussed in chapter III and in the first part of chapter IV, could not deal with these problems at all. As the reader will remember, both the capital accumulation conditions and the effective demand condition, in those models, have been assumed to be independent of time and of the stage of development the system has reached, so that — once defined at a certain point of time — they remained the same for all the time. This meant giving up the possibility of investigating any type of structural dynamics, and at the same time it meant frustrating the purpose itself of a disaggregated formulation.

The whole picture becomes radically different in the present model, where the fulfilment of the equilibrium conditions at time zero is no longer the end but just the beginning of the whole story. These conditions *cannot* remain the same as time goes on; because technical progress — whether uniform over the whole economy or not — causes each single component of the summations in (V.11) to change. This means that the way in which each one of conditions (V.8) and (V.11) are fulfilled must be continually different as time goes on. The size of the various sectors that may satisfy those conditions in a given period of time is necessarily different from the size of the same sectors which ensured their fulfilment in the previous period, and again is necessarily different from the size of the same sectors which can ensure their fulfilment in the following period.

But the discussion of the previous chapter now allows us to go far beyond these general remarks. By hypothesis, almost all technical coefficients are decreasing in time. This means that, unless the demand coefficients increase in the same proportion, condition (V.11) is bound to become under-satisfied as time goes on. But we know already that no demand coefficient can increase indefinitely, because eventually all demand coefficients reach saturation level. Therefore we must conclude that condition (V.11), as it stands, inevitably manifests a tendency to become under-satisfied, i.e. to generate unemployment, as time goes on.

Fortunately there are two factors, operating in the long run, which come to counterbalance the above mentioned tendency. These two factors must now be introduced into the model. One of them is the same one which causes the whole trouble: technical progress. So far in this chapter technical progress has been considered in the form of increases of productivity. But it has been pointed out earlier that technical progress also takes the form of introducing new goods. Our model must therefore be completed now by opening it to the possibility of the introduction of new sectors. This can be done by taking the number of sectors as running no longer from 1 to a fixed $(n-1)$, but from 1 to let us say $x(t)$, where $x(t)$ denotes a number which increases as an effect of technical progress. Thus condition (V.11) must be written:

$$\text{(V.12)} \qquad \sum_{i=1}^{x(t)} a_{ni}(t-\theta)\, a_{in}(t-\theta)\, e^{(r_i-\varrho_i)\theta} + \sum_{i=1}^{x(t)} \left(\frac{1}{T_i}+g+r_i\right) a_{nk_i}(t-\theta)\, a_{in}(t-\theta)\, e^{(r_i-\varrho_{k_i})\theta} = \alpha(t)\, \beta(t) \,.$$

All this means that the effects of technical progress on the effective demand condition are twofold. On the one hand, it brings about an average decrease through time of the $a_{ni}a_{in}$'s referring to the commodities so far produced, and on the other it keeps on *adding* new coefficients (referring to new commodities). The second tendency may succeed in counter-balancing the first.

However, if this counter-balance still does not come about, there is a second way in which condition (V.12) may be kept satisfied in time. That is through a long-run diminishing trend of the right hand side of the equation; namely of the coefficients α and β. This means a decreasing trend of the working time (i.e. an increasing trend of leisure time), to be achieved by a decrease either of the proportion of active to total population or of the length of the working week.

All these conclusions may seem, after all, to boil down to the common-sense proposition that technical progress gives society a choice between producing more or new goods, and enjoying more leisure. Our analytical formulation, however, just because of its macro-economic implications, reveals something more than that; it evinces the fixed framework within which the choice has to be made. It shows that there is a third element in the problem — the requirement of keeping full employment — which restricts the choice to those combinations of goods and leisure which, in terms of time required, add up to a fixed amount determined by the existing working population and technical knowledge. To express the same thing in another way, we might say that the choice between products and leisure is not merely a possibility, but a necessity, if full employment is to be maintained. (There does not exist the alternative of not choosing). Technical progress, which characterises our societies and which brings with it that choice, does not come — so to speak — under the form of a gift which, by being always susceptible of being refused, only could add to, and never diminish, the pre-existing wealth. It comes under the form of a flow, which cannot be stopped and has to be continually channelled in new directions, which themselves have to be discovered anew because the old ones saturate. This entails an ever-standing problem of utilization, under a fixed restriction represented by condition (V.12); a problem for which the failure to find solutions may cause damage to the previous situation.

Before closing our comments on condition (V.12), we may add that there is a third way of keeping equilibrium over time. This would be to regard — as indeed we have done — the variations of the r_i's as determined by the evolution of consumers' preferences, but to have among them at least some r_i's which are not dependent on such preferences and which are flexible enough and capable of being influenced in such a way as to keep condition (V.12) satisfied. This is not a fanciful remark; it is in fact an important one, on practical grounds. It means that, if there is an external Agent or institutional Organization that is interested in keeping full employment, it is possible for this Agent or Organization — when any other mechanism which might have been put into operation fails — to act and to attain the aim of full employment simply by inflating total demand.

6. *The dynamic movements of physical quantities and of relative prices*

When conditions (V.8), (V.12) are satisfied, each of the two linear and homogeneous systems of equations representing the flows of the economy — the (II.9) and (II.13) — yields solutions for all its unknowns but one, which can be arbitrarily fixed. But since, under our present set of hypotheses listed in section 1, all coefficients and unknowns are dated, the solutions no longer take the form of single values but that of *movements through time*. It is evidently one of these movements which, in each of the two systems, can be arbitrarily fixed. In the case of system (II.9) there is already one of the X's — namely X_n: population — whose movement has been accepted as given from outside economic analysis. On the other hand, in the case of system (II.13), no one of the prices is given so that the structural dynamics of the system only determines the movements of relative prices. As a matter of convenience, we may take the wage rate as given through time ($\bar{W}$), so that the dynamic movements of physical quantities and of relative prices emerge as follows:

$$\text{(V.13)} \qquad \begin{cases} X_i(t) = A\, e^{(g+r_i)\theta}\ , \\ X_{k_i}(t) = A \left(g + \frac{1}{T} + r_i \right) e^{(g+r_i)\theta}\ , \end{cases}$$

$$\text{(V.14)} \qquad \begin{cases} P_i(t) = B\, e^{-\varrho_i \theta} + C \left(\frac{1}{T} + \pi \right) e^{-\varrho_{k_i}\theta}\ , \\ P_{k_i}(t) = C\, e^{-\varrho_{k_i}\theta}\ , \end{cases}$$

$$i = 1,\ 2,\ \ldots\ x(t),$$

where A, B, C are constant which stand for the initial conditions, namely

$$
\begin{aligned}
A &= a_{in}(t-\theta)\ X_n(t-\theta),\\
B &= a_{ni}(t-\theta)\ \overline{W},\\
C &= a_{nk_i}(t-\theta)\ \overline{W}.
\end{aligned}
$$

The economic meaning of (V.13), (V.14) *at a given point of time* has been discussed already on the occasion of our short-run analysis of chapter II (sections 5 and 6), and there is no need for repetition here. But besides the determination of quantities and prices at a given point of time, the (V.13), (V.14) now bring out their movements through time as well.

As can be seen, each physical quantity follows a time-path of its own in time, expanding at a particular rate $(g+r_i)$ which is the sum of two rates of increase: the rate of growth of population and the rate of change of *per-capita* demand. The first of these two rates is the same for all goods, but the second is different from one commodity to another. Therefore, unless *per-capita* demand remains constant in time (i.e. unless there is no technical progress) *the whole production structure in physical terms is changing as time goes on*. This means that, while growing, the system is continually changing the proportions in which it produces the various commodities.

The (V.14) show moreover that each relative price is also changing in time in its own way. With the convention of adopting W, the wage rate, as *numeraire*, each price is decreasing at a rate resulting from a weighted average of the pace at which technical progress is going on in the sector to which it refers, and in the sector which produces capital goods for it. Since all these rates are also different from one another, *the whole price structure is changing as time goes on*.

7. *The dynamic movements of the other variables*

Following the procedure which has been used in chapter III, the movements through time of other magnitudes of economic interest (like sectoral production at current prices, sectoral employment, capital-output ratios, etc.) can be derived as simple corollaries from (V.13) and (V.14). These other economic magnitudes depend both on (V.13) and on (IV.14), i.e. both on technology and on demand. Therefore their dynamic movements result from a composition of the movements through time of physical quantities and of relative prices. Let us consider them in detail.

1. The time-paths of the production of each commodity evaluated at current prices [the $V_i(t)$ and $V_k(t)$] can evidently be obtained by multiplying each of the (V.13) by each of the (V.14). The result is

$$(V.15) \qquad \begin{cases} V_i(t) = D\, e^{(g+r_i-\varrho_i)\theta} + G\left(\pi + \frac{1}{T}\right) e^{(g+r_i-\varrho_{k_i})\theta}, \\ V_{k_i}(t) = G\left(g + \frac{1}{T} + r_i\right) e^{(g+r_i-\varrho_{k_i})\theta}, \end{cases}$$

$$i = 1, 2, \ldots x(t),$$

where the constants D and G stand for the initial conditions, namely

$$D = a_{ni}(t-0)\; a_{in}(t-0)\; \overline{W}\; X_n(t-0),$$

$$G = a_{nk_i}(t-0)\; a_{in}(t-0)\; \overline{W}\; X_n(t-0).$$

As can be seen, the V_i's and the V_{k_i}'s change in time for two reasons: because the physical quantities are changing and because prices are changing. Having taken the wage rate as *numeraire*, the (V.15) show that each rate of change of these variables comes out as an algebraic sum of the rate of population growth, the rate of change of *per-capita* demand and the rate of increase of productivity (the latter with the negative sign).

2. Similarly, the dynamic movements of employment in the various sectors emerge as follows:

$$(\text{V.16}) \qquad \begin{cases} E_i(t) = M\, e^{(g+r_i-\varrho_i)\theta}, \\ E_{k_i}(t) = N\left(\frac{1}{T}+g+r_i\right) e^{(g+r_i-\varrho_{k_i})\theta}, \end{cases}$$

$$i = 1, 2, \dots x(t)),$$

where (5):

$$M = a_{ni}(t-\theta)\; a_{in}(t-\theta)\; \frac{1}{\alpha}\, X_n(t-\theta),$$

$$N = a_{nk_i}(t-\theta)\, a_{in}(t-\theta)\, \frac{1}{\alpha}\, X_n(t-\theta).$$

In words, employment in each sector i moves through time at a rate of change equal to the rate of population growth plus the rate of increase in *per-capita* demand for commodity i, minus the rate of increase of productivity in the sector.

(5) The $E_i(t)$ and $E_{k_i}(t)$ have been expressed here in terms of men per unit of time. For example, if the unit of time is one year, they would be expressed in terms of man-years; so that:

$$\Sigma[E_i(t) + E_{k_i}(t)] \quad = \alpha\, X_n(t).$$

Here we have new problems arising because the E_i's and E_{ki}'s although representing flow-variables (services from labour in the unit of time) come from, and are inseparably linked to, a stock-variable (the labour force) which may not be perfectly mobile. As appears from (V.16), if population is constant ($g=0$) employment in each sector i increases or decreases through time according to whether $r_i > \rho_i$ or $r_i < \rho_i$. This means that, on the average, half of the sectors are offering jobs and half of the sectors are dismissing workers, as time goes on. Clearly, this may be a very serious state of affairs — especially in a very progressive system, i.e. in a system with very high ρ_i's — if labour is highly specialized and is not susceptible of being transferred from one sector to another except at the expense of heavy losses in productivity.

Fortunately, the natural movements of population come in here to help in the right direction and in two senses. First of all, the natural process of people ageing permits a redistribution of employment among sectors by addressing young workers towards expanding sectors and by not replacing retired people in the contracting sectors. This may be a slow process, but it is one which is going on even when population is stationary. Secondly, when population is growing, its rate of increase is a net positive addition to the rate of change of demand (and therefore of employment) in *all* sectors. It follows that, in absolute terms, only those sectors will actually lose employment where the rate of increase in productivity is so high and the rate of increase in *per-capita* demand so low that the difference between the two is not only negative, but negative to such an extent as to make the sum $(g + r_i - \rho_i)$ less than zero, or rather the sum $(g + r_i - \rho_i + \delta)$ less than zero, where δ stands for the rate of people's retirement from working activities [6].

[6] At this point restrictions (V.6), discussed in section 2, may be re-written with reference both to the stocks of capital and to employment of labour in each single sector, in the following way:

$$g + r_i + \frac{1}{T_i} \geq 0, \qquad \text{(V.11)}$$

(continued on following page)

Agriculture seems to be one of the most typical sectors of this kind.

To sum up, we may say that, as time goes on, the whole structure of employment changes, in the sense that *the proportions* in which total employment is distributed among the different sectors of the system are changing. However, actual dismissal of workers from some sectors will take place only in those cases where the increase in population is not enough to counter-balance the effect of a decreasing *per-capita* demand or of an increasing productivity, or of the algebraic sum of the two. This means that the higher the rate of population growth (when the capital accumulation needed to absorb it can easily be afforded), the easier it is for an economic system to adapt itself to a given structural process of change of employment.

3. The movements through time of the sectoral capital-output ratios also follow straightforwardly from the previous analysis, namely from expressions (III.14) and from the dynamic movements (V.13) and (V.14). Here they can perhaps be better examined by considering their reciprocals, which emerge as follows:

$$\text{(V.17)} \qquad \frac{1}{\varkappa_i(t)} = \frac{1}{T} + \pi + \frac{a_{ni}(0)}{a_{nk_i}(0)} e^{(\varrho_{k_i}-\varrho_i)t} , \qquad i = 1, 2, \dots x(t).$$

$$\text{(V.2n)} \qquad g + r_i - \rho_i + \delta \geq 0, \qquad i = 1, 2, \dots x(t).$$

We may say that, when all (V.1n)-(V.2n) are satisfied, the model retains all its properties. But if some of these inequalities should be reversed, then in order to tell what will happen one needs more information about the degree of flexibility of capital — in the case of V.1n) — and about the degree of mobility of labour — in the case of (V.2n). If no mobility of capital or of labour is possible between one sector and the others, then some idle capacity — in the case of (V.1n) — and some technological unemployment — in the case of (V.2n) — are bound to appear.

The first point to make about the (V.17) is that if T and π remain constant, the sectoral capital-output ratios exclusively depend on technology, both at a certain point of time and in their movements through time. As time goes on, each of them increases, decreases or remains constant according to whether the rate of increase of productivity in the sector considered is higher than, lower than, or equal to, the rate of increase of productivity in the corresponding capital goods sector. The reader will recognize in these three possibilities the cases which in dynamic economics [7] are commonly known as the cases of *labour-saving*, *capital-saving*, and *neutral* technical progress. We can only confirm here that, with reference to the *sectoral* capital-output ratios, this classification is perfectly justified, because the variations can indeed be traced back to a particular type of change in technical knowledge.

4. But let us now consider the aggregate capital-output ratio. Again, it becomes convenient to look at its reciprocal (the output-capital ratio). By substituting (V.13)-(V.14) into (III.15) and by calling r the over-all average rate of growth of *per-capita* demand (i.e. the weighted average of the rates of increase of total demand in each single sector), the over-all capital-output ratio emerges as

$$\text{(V.18)} \qquad \frac{1}{\chi(t)} = \left(\frac{1}{T} + \pi + g + r\right) = \frac{\sum a_{in}(t-\theta)\, a_{ni}(t-\theta)\, e^{(r_i-\varrho_i)\theta}}{\sum a_{\cdot n}(t-\theta)\, a_{nk_i}(t-\theta)\, e^{(r_i-\varrho_{k_i})\theta}},$$

Now, the most remarkable difference that $\chi(t)$ exhibits with respect to each of the $\varkappa_i(t)$ is that it depends not only on technology, but also on *demand*. As can be seen by a simple

[7] These definitions are due to R.F. HARROD who foreshadowed them already in his review of JOAN ROBINSON'S, *Essays in the Theory of Employment*, « The Economic Journal », 1937, pp. 328-9, and finally formulated them in *Towards a Dynamic Economics*, London, 1948, pp. 22-23.

comparison, expressions (V.17) only contain technical coefficients and their rates of change over time; while expression (V.18) contains, besides these coefficients and their rates of change, also the *demand* coefficients and their rates of change over time.

There is one immediate conclusion that may be drawn. It is not permissible to talk of neutral, capital-saving and labour-saving technical progress merely on the basis of changes in the over-all capital-output ratio, because technical progress is only one of the two factors on which the over-all capital-output ratio depends. It would be quite possible, for example, to envisage a case in which technical progress is capital-saving in each single sector of the economy and nevertheless the aggregate capital-output ratio remains constant or even increases as time goes on, simply because *per-capita* demand is expanding in the direction of highly capital-intensive commodities; and *vice versa*. This also means that all recent discussions on the factors affecting the aggregate capital-output ratio, discussions which have stemmed from macro-economic models and have focussed only on technology, have in fact missed one half of the problem. As emerges from the foregoing analysis, any explanation of the movement in time of the aggregate capital-output ratio cannot be correct, if it does not consider both sides of the problem: the demand side as well as the technical side.

5. A few final remarks may be added about the dynamics of the aggregate variables normally used in macro-economic investigations (i.e. national income, total capital, consumption, investment, etc.). From the (V. 13)-(V-14) it follows that all these aggregate magnitudes move in time in a very composite way. Each of them results from a sum of physical quantities, whose proportions are changing, multiplied by prices which also change as time goes by. Therefore, all of them have a well defined meaning at any specific point of time, in relation to the

technical and demand conditions prevailing at that point. But owing to the simultaneous change of the two component structures (physical quantities and relative prices), their comparisons through time entail the so-called « index-number problem ». The difficulties arising from this problem may be negligible when comparisons refer to short periods of time, but they increase more and more as the time elapsing between the aggregate magnitudes to be compared becomes longer and longer.

8. *Short-run flexibilities*

Before closing the chapter, it may be useful to hint briefly at short-run possibilities of adjustments. If displaced from positions of long-run equilibrium, the system possesses in the short run, some important possibilities of flexible adjustments, owing to the parameters α, β and T.

The values of α, β, T which enter the model represent the long-run normal values of these parameters, which may be constant or showing mild trends in time. But in the short-run these parameters can be easily and widely influenced, thereby affecting the size of the stocks that the system can utilize. This means that productive capacities and labour never represent absolutely rigid concepts, at least in the same way as the physical equipment and the population existing at any certain time do. Between these latter physical concepts and those parts of them which are relevant to the productive process, the mentioned parameters provide a sort of flexible cushion. Temporary variations of β and T mean temporary increases in the productive capacities of existing physical plants by utilizing them more intensively or by keeping in operation some machines which were due to be scrapped. Similarly, variations of β and α mean increases of working-hours out of a given population, obtained by lengthening the working week or by increasing the proportion of active population (more working women or later retirement ages).

These flexibilities evidently represent very important factors in allowing the system the possibility of adjustment and of keeping stability in the short run, in face of disturbances of various kinds.

Chapter VI

The empirical significance of the model

1. *The relation of sectoral dynamic analysis to input-output analysis*

The model which has been developed in the previous pages has far-reaching theoretical implications and also straightforward empirical applications. Leaving its theoretical implications aside for the time being, we may concentrate in this chapter on its empirical significance. At the same time, the opportunity will be taken of completing the model with reference to inter-industry relations at a given point of time. It will be remembered that, in Chapter II, all inter-industry connections and intermediate commodities at a given point of time were deliberately left aside, because our aim was to arrive at a dynamic investigation as soon as possible. We may now go back for a moment to that stage. Fortunately, there is no need to develop here any model with intermediate commodities; for such models have been extensively developed already in the economic literature, especially during the past twenty years. Hence our task can be limited to showing how they relate to the previous analysis.

There are in particular two theoretical schemata, recently presented, which may be considered as the logical static counterpart of the previous dynamic analysis. They are Wassily Leontief's input-output model, and Piero Sraffa's production system (1). The latter perhaps corresponds better than the former to the approach taken in the present work, but the former has had in practice wider empirical applications. Therefore, it will be convenient here to take LEONTIEF's system as the static term of our comparisons.

First of all, let me point out the similarity of approach, from an empirical point of view, of the previous dynamic model and the static input-output model. Both models share the characteristic of being built on coefficients which represent actual outcomes and which can, therefore, at least in principle, be given an empirical content simply by recording the actual performance of the economic system. Of course, the technical and the consumption coefficients, by which this actual performance is represented, come from a choice, made from among a larger set of possibilities. But all the alternative possibilities that might have, but have not, been chosen have become irrelevant. The coefficients that appear both in the input-output and in the present model must, therefore, be interpreted as simply representing those real quantities which can actually be observed.

Let us notice, moreover that LEONTIEF's system and the present model also coincide in the way they look at the final sector of the economy — the last column of the coefficient matrices are the same in the two systems (with the only difference that, in the present model, consumption goods and investment goods are listed separately). However, they differ profoundly in the way they consider the production processes. The same production structure of the economy is looked at from two different points of view — one is very close to it and to what is immediately observable; the other is placed much further away, at the final stage of the consumption and investment goods. The LEONTIEF approach can certainly be more

(1) WASSILY W. LEONTIEF, *The Structure of American Economy*, 1919-1939, New York, 1941 and 1951; PIERO SRAFFA, *Production of Commodities by means of Commodities*, Cambridge, 1960.

immediately grasped. One of the things by which one is most impressed, when looking at the real transactions which take place in an economic system, is the great number of interrelations among productive units. One's first instinct is, therefore, to inquire into these inter-industry connections and try to reproduce them analytically. This is the idea which already underlay QUESNAY's *tableau economique* and which has been developed and given a full empirical content by LEONTIEF.

A different approach is taken in the present model. Not « industries », in the input-output sense, but « sectors » are taken as the basis of the whole investigation. And sectors are defined in such a way as to be *vertically integrated*. All interrelations which can be observed in the real world are looked at as parts of a process which has not yet come to an end. Any process reaches its completion only when the product which comes out is a final commodity (consumption or investment goods). A vertically integrated sector is, therefore, from an inter-industry point of view, a very complex one as it goes through and through the whole intricate inter-industry connections. However, from the point of view of the homogeneity of the inputs, it becomes a very simple one, as it eliminates all intermediate goods and resolves each final commodity into its ultimate constituent elements: a (flow) quantity of labour and a (stock) quantity of capital. It may be interesting to recall that the procedure has already been used by LÉON WALRAS in his *Elements of Pure Economics,* although in a more rudimentary way (2).

At a given point in time, between the two ways of looking at the economic system there is really no logical difference. Both models represent the same thing, looked at in a different way. The difference, in other words, lies only in the classification, and we can pass from the one to the other simply by an

(2) See p. 241 of the English edition of LÉON WALRAS's *Elements d'économie politique pure,* translated, collated and edited by W. JAFFÈ, Homewood (Ill.), 1953.

algebraical re-arrangement, corresponding to a process of solving a system of linear equations: the production coefficients of one model turn out to be a linear combination of the production coefficients of the other.

This can be shown immediately, if goods are expressed in physical terms. (A further similar algebraical re-arrangement would then be needed for the investment goods if they are to be expressed in terms of physical capacities). Thus if we take an input-output system, we must state consumption goods industries and investment goods industries separately. We can then isolate the inter-industry transactions by opening the system with respect to the *final* sector. This means that we take as given the final demands, common to both LEONTIEF's and the present model, and drop from the system the last row, representing the inputs of the original factor (labour) into each industry.

We obtain:

$$\text{(VI.1)} \qquad \begin{bmatrix} 1-c_{11} & \cdot\;\cdot & -c_{1j} & \cdot\;\cdot & -c_{1,n-1} \\ \cdot & \cdot\;\cdot & \cdot & \cdot\;\cdot & \cdot \\ -c_{i1} & \cdot\;\cdot & 1-c_{ij} & \cdot\;\cdot & -c_{\;,n-1} \\ \cdot & \cdot\;\cdot & \cdot & \cdot\;\cdot & \cdot \\ \cdot & \cdot\;\cdot & \cdot & \cdot\;\cdot & \cdot \\ -c_{n-1,1} & & -c_{n-1,j} & & 1-c_{n-1,n-1} \end{bmatrix} \begin{bmatrix} Z_1 \\ \cdot \\ Z_i \\ \cdot \\ \cdot \\ Z_{n-1} \end{bmatrix} = \begin{bmatrix} U_1 \\ \cdot \\ U_i \\ \cdot \\ \cdot \\ U_{n-1} \end{bmatrix}$$

where the c_{ij}'s stand for the inter-industry technical coefficients, the Z_i's for the productions of intermediate commodities, and the U_i's for the final demands ($i,j = 1, 2, \ldots, n-1$). By solving (VI.1), we arrive at

$$\text{(VI.2)} \qquad \begin{bmatrix} Z_1 \\ \cdot \\ Z_i \\ \cdot \\ Z_{n-1} \end{bmatrix} = \begin{bmatrix} 1-c_{11} & \cdot & -c_{1j} & \cdot & -c_{1,n-1} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ -c_{i1} & \cdot & 1-c_{ij} & \cdot & -c_{i,n-1} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ -c_{n-1,1} & & -c_{n-1,j} & & 1-c_{n-1,n-1} \end{bmatrix}^{-1} \begin{bmatrix} U_1 \\ \cdot \\ U_i \\ \cdot \\ U_{n-1} \end{bmatrix},$$

where the superscript -1 indicates the operation of matrix inversion. Now each column of the inverted matrix represents the amounts of all intermediate goods which have gone into one unit of final commodity. This means that by multiplying each column of the inverted matrix by the row of the input-output labour coefficients which has been excluded from (VI.1), we arrive at the labour coefficients of the vertically integrated system.

In algebraic terms:

$$\text{(VI.3)}\quad \begin{bmatrix} a_{n1} \\ \cdot \\ \cdot \\ a_{nj} \\ \cdot \\ \cdot \\ a_{n,n-1} \end{bmatrix} = \left\{ \begin{bmatrix} 1-c_{11} & \cdot & -c_{1j} & \cdot\ \cdot & -c_{1,n-1} \\ \cdot & & \cdot & \cdot\ \cdot & \cdot \\ \cdot & & \cdot & \cdot\ \cdot & \cdot \\ -c_{i1} & & 1-c_{ij} & & -c_{i,n-1} \\ \cdot & & \cdot & \cdot\ \cdot & \cdot \\ \cdot & & \cdot & \cdot\ \cdot & \cdot \\ -c_{n-1,1} & & -c_{n-1,j} & & 1-c_{n-1,n-1} \end{bmatrix}^{-1} \right\}' \begin{bmatrix} c_{n1} \\ \cdot \\ \cdot \\ c_{nj} \\ \cdot \\ \cdot \\ c_{n,n-1} \end{bmatrix},$$

where the mark ' denotes the operation of matrix transposition. Equations (VI.3) now give the algebraic relation, existing in a given period of time, between the labour coefficients of an input-output model and the labour coefficients of the vertically integrated sectors used in the present dynamic model. The ones may be obtained from the others — as the (VI.3) now directly show — by a straight-forward algebraical operation.

2. *Fitting empirical data into the model*

The algebraic relation which has just been obtained and which links the input-output technical coefficients with the technical coefficients of our analysis finds an immediate application in the problem of giving the various symbols so far used an empirical content. Of course, since each coefficient or variable appearing in our system refers to actually observable magnitudes, the problem of fitting empirical data into the model is in principle very straightforward, in the same way as it is for an input-output model. But the criterion of classification is different. In the input-output model, the criterion is the industry producing a certain commodity, intermediate or final, and the problem is to reckon where its inputs come from and where its outputs go to. In the present model, the criterion is the process of production of a *final* commodity, and the problem is to build behind each final commodity a conceptually integrated sector which, by passing over all the intermediate commodities, goes right back to the original factors.

The procedures for collecting and ordering data for input-output analysis purposes are well known. On the other hand, collecting data for the purposes of the present model may seem a rather laborious task, mainly in connection with the production sectors, if not in connection with the final sector (the latter being the same as the final sector of an input-output model). But the difficulties are only apparent. It is true that to classify production processes in a vertically integrated way would be almost an impossible task if attempted directly. But this task need not be attempted directly. The algebraical procedure which has just been shown allows us to go over to a vertically integrated type of classification by starting from a classification of the input-output type.

All this means that, in order to give the present model an empirical content, one may first of all collect data and fit them into an input-output table in the usual inter-industry way. Moreover, data must be collected about capital at current prices (or about capital-output ratios, as is more usual) in each single industry of the input-output classification. Then the resulting system of input-output linear equations can be *open* with respect to the final goods and solved by computing the inverse of the coefficient matrix. This inverse matrix, or more precisely the transpose of this inverse matrix, as shown in the previous section, provides the link between the input-output type of classification and the one which is needed in the present model. This means that to pass from one classification to the other is simply a matter of computation. The procedure, which has been shown by (VI.3) with reference to the labour coefficients, remains exactly the same for the stocks of capital (or for the series of capital-output ratios). In other words, after multiplying the transposed inverse matrix by the vector of the capital stocks (or of the capital-output ratios) of each input-output industry, we obtain the vector of the capital stocks (or of the capital-output ratios) of each vertically integrated sector. The transposed inverted matrix appears, therefore, as the linear operator which may be applied to a classification of labour and capital according to the inter-industry relations, in order to reclassify it according to a new type of vertically integrated sectors.

In this way, each production process is reduced to one flow-input — labour — and one stock quantity—capital. The coefficients representing them do not correspond to labour or capital employed in any particular firm or industry, since the whole framework of intermediate relations has been consolidated; but they do represent all the labour and capital which are necessary to produce the commodity under consideration, in whatever remote corner of the economy they have been applied. Formally, the new coefficients are, therefore, derived concepts (derived from the consolidation of the inter-industry coefficients) but they have a deeper economic meaning and, as will be stressed in a moment, possess more favourable characteristics for a dynamic analysis.

3. *The rationale of framing a dynamic analysis in terms of vertically integrated sectors*

At this point, the reader may wonder why, in the previous dynamic analysis, a classification based on vertically integrated sectors has been preferred to an input-output type of classification. This question can be answered simply by considering the type of analysis for which each of the two classifications is most suited.

To begin with, I may recall that both the inter-industry and the vertically integrated way of looking at the production processes of an economic system are by no means new in economics; they can be found quite extensively used at different stages in the history of economic thought. However, it is very significant that they have normally been used for different purposes and independently of each other: the inter-relation approach has mostly been associated with analysis at the micro-level and of a static nature, while the vertically integrated approach has mostly been associated with dynamic and macro-economic types of investigations. As a result, a kind of gap has gradually appeared between the two approaches. The foreigong discussion now puts us in a position to investigate the nature of this gap.

At a given point of time, the two models which we have been confronting make the connections between the two approaches quite obvious and well-defined. LEONTIEF has provided, for the inter-relation approach, a much more aggregate framework than the one normally used. The model which has been discussed here gives, on the other hand, a much more disaggregated framework for the vertically integrated approach. Between the two, the inverse matrix mentioned above provides the analytical bridge. As a matter of fact, once we posses the inverse matrix, all relations between the two approaches at a given point of time take the form of one-to-one correspondence. No gap really exists in this case: the two ways of looking at the production activities meet half-way, through the above mentioned inverse matrix, which represents the analytical tool for re-classifying the same transactions according to two different criteria and points of view (3). Of course, the input-output model gives us more information. If we were simply interested in what happens at a specific point of time, the input-output model would be the obvious one to use because it provides a more complete picture.

But as time goes on, the input-output coefficients change and the inter-industry system breaks down. The connections described above begin to vanish. Then it is only the vertically integrated model that allows us to follow the vicissitudes of the system through time. It may perhaps be useful to point out explicitly how this happens. The process of technical change, as has been remarked earlier, manifests itself in a continuous way at the level of the single units of production, especially in the form of slow improvements, coming from patiently trying different raw materials, re-thinking the disposition of the line of production, eliminating bottle-necks in the production process, trying new similar products, etc.. Even when new methods of production or new products are invented, their introduction into the economic system very rarely takes the form of a sudden change. Most of the time, the new products or the new methods are operated uneconomically for a period, until experience and slow improvements put them on a competitive footing and prepare the ground for further improve-

(3) Professor LEONTIEF himself has, on a couple of occasions, adopted the vertically integrated type of classification. See especially *Domestic Production and Foreign Trade: the American Capital Position Re-Examined*, « Proceeding of the American Philosophical Society », vol. 97, No. 4, Sept. 1953, and *Factor Proportions and the Structure of American Trade: Further Theoretical and Empirical Analysis*, « Review of Economics and Statistics », Nov. 1956.

ment and development. To an external observer, looking for the « best-known technique », the methods of production may sometimes look like being introduced at a certain point and then remaining stable for long periods of time; but when a reckoning of the inputs and outputs of an industry is begun, the most widely warying shifts are found from one moment to the next. Even at a given point of time, there are many directions from which the inputs may come, and to which the outputs may go.

These are well-known problems which have always caused difficulties to all builders of input-output tables and which sometimes cast serious doubts on the meaning to be attributed to the coefficients of very disaggregated input-output systems. Of course these doubts diminish the more the industries are aggregated, but in this direction the meaning of an input-output framework diminishes too. So that input-output experts have always tried to find — so to speak — a sort of *minimax* point at which to stop the process of aggregation-disaggregation, in such a way as to keep the usefulness of an input-output table without making its coefficients too unstable.

What has not been sufficiently realized is that the property of eliminating these shortcomings belongs to the completely aggregated quantities not because they are aggregated but because — by being completely aggregated — they are necessarily vertically integrated. The property extends to all vertically integrated magnitudes as well. By resolving all products only into the same constituent original elements — labour and capital — the vertically integrated approach leads to setting up relations whose permanence over time does not depend on the different technical possibilities. For example, two equivalent methods which, at a given point of time, entail the same cost for the same output, are represented in an inter-relation system by *two different* technical functions. But in a vertically integrated system, they are expressed by exactly the same function. Their being equivalent means that they require the same amount of inputs, and these are expressed in the same terms independently of the industry in which they are used. Similarly, a sudden shift taking place at the level of a particular technical process in the origin of one of the inputs (for example a shift in the provenance of a fibre from the textile industry to the chemical industry) means that less labour than before is required to produce it through the new channel. A shift of this type changes an inter-industry relation by causing the disappearance of a coefficient (and of the correspondent variable) and the appearance of another, different, one. In the vertically integrated relation, it only causes a small diminution of *the same* coefficient. If we imagine many of these shifts taking place while time goes on and, along with them, corresponding changes in the consumption coefficients — which is, after all, the normal path that technical progress takes — the input-output table is continuously upset and all functions change from one moment to the next. On the other hand, the vertically integrated relations remain unaffected. The only consequence for them is that their coefficients gradually diminish through time in a movement which, for analytical purposes, has been approximated in the present model by a smooth trend developing at a certain rate of change.

Concluding and summarising, we may say that, at any given point of time, there exists between the static input-output model and the model presented in the previous pages a very definite relation through a fully specifiable matrix of coefficients. Considered at a given point of time, the input-output model is more analytical — it has much more to say about the structure of an economic system. However, as time goes on, and the conditions of production and of consumption change (owing to technical progress, economies and diseconomies of scale, etc.) the inter-industry relations break down and become different from one moment to the next, so that a particular input-output table is needed for each stage in the evolution of the economy under consideration. These tables can be compared (comparative statics analysis), but they cannot be analytically linked to one another — no theory of any generality can be provided for passing from the one to the other (4). The continuity in time is kept, on the other hand, at the vertically integrated level, where the relations which can be set up possess — to use FRISCH-HAAVELMO's terminology (5) — a higher degree of autonomy. This means that the permanence of these relations in time is independent of technical change. In this context, the vertically integrated technical coefficients acquire a meaning of their own, independent of the origin of the single parts which compose them. The movements of these coefficients through time, and the various consequences thereof, can be investigated and followed as such. When more information is needed about the

(4) A note may be added here about how this applies to the work which is being done at present at the Department of Applied Economics of Cambridge, where Professor STONE, Mr. BROWN and their colleagues are working on a model of economic growth for the U.K. from 1960 to 1970. They seem to be trying to make an estimate of the 1970 input-output table by applying uniform coefficients of reduction to the rows of the table for 1960. (See: RICHARD STONE and ALAN BROWN, *A Computable Model of Economic Growth*, D.A.E. Cambridge 1962, especially pp. 70-71).

The procedure, as such, hardly has any theoretical justification if technical progress follows the pattern which has been described in the text. Yet, since the period considered is not too long and since the table adopted is rather aggregate (of the order of 30 industries), which means that the industries considered are not very far from being vertically integrated sectors, the results obtained thereby may turn out to be not too unsatisfactory after all.

One should realize, however, that whatever degree of satisfaction there may be in the results, it cannot be attributed the procedure, which is unacceptable in principle. It is to be attributed to (and will be greater, the greater the degree of) shortness of the period considered, and aggregation, i.e. vertical integration, of the industries considered.

(5) TRYGVE HAAVELMO, *The Probability Approach in Econometrics*, Suppl. to « Econometrica », July 1944. The process of passing from inter-industry to vertically integrated relations for the purpose of dynamic analysis seems to be a typical example of what HAAVELMO describes as a way of passing to more fundamental and autonomous relations. « In scientific research, our search for *explanations* consists of digging down to more fundamental relations than those that appear before us when we merely *stand and look*. Each of these fundamental relations we conceive of as invariant with respect to a much wider class of variations than those particular ones that are displayed before us in the natural course of events » (*ibid.*, p. 38).

industrial structure at a particular point of time, the vertically integrated coefficients can be split and analysed into inter-industry coefficients particular to that point of time.

In this way the static input-output and the dynamic vertically-integrated systems appear as mutually complementary and completing each other. Inter-industry relations, referring to any particular point of time, represent a cross-section of the vertically integrated variables, whose movements through time express the structural dynamics of the economic system.

Appendix to Chapter VI

A Criticism of the von Neumann Type of Dynamic Models

The vertically-integrated model developed in the present work — as the reader has by now realised — has taken so marked a departure from all the dynamic extensions of the inter-industry system, which have become so widespread in current economic literature, as to require perhaps some justification. To this purpose, I may append a critical assessment of the approach adopted in all the dynamic models which are currently discussed among mathematical economists.

As is well known, the most outstanding of all of them is von Neumann's extension over time of the general equilibrium system (1). Another model of particular interest to us is, of course, Leontief's dynamic version of his static input-output analysis (2). All the other works that have followed (3)

(1) John von Neumann, *A Model of General Equilibrium*, « The Review of Economic Studies », 1945-46.

(2) W.W. Leontief, *Dynamic Analysis*, Chapter 3 of « Studies in the Structure of the American Economy », by W.W. Leontief and others, New York, 1953. This dynamic analysis of Leontief's has been anticipated by David Hawkins, *Some Conditions of Macroeconomic Stability*, « Econometrica », 1948.

(3) See, for example: Kemeny, Morgenstern and Thompson, *A Generalization of the von Neumann Model of Expanding Economy*, in « Econometrica », 1956; D. Gale, *The Closed Model of Production*, in « Linear Inequalities and Related Systems », Annals of Mathematical Study, No. 38, ed. by H.W. Kuhn and A.W. Tucker, Princeton, 1956; R. Dorfman, P.A. Samuelson, R.M. Solow, *Linear Programming and Economic Analysis*, New York, 1958, chapters 11 and 12; M. Morishima, *Prices, Interest and Profits in a Dynamic System*, « Econometrica », 1958; *Some Properties of a Dynamic Leontief System with a Spectrum of Techniques*, « Econometrica », 1959; *Economic Expansion and the Interest Rate in Generalized von Neumann Models*, « Econometrica », 1960.

have introduced many slight variations in these two models but have left unchanged their basic features.

Let me begin by considering VON NEUMANN's celebrated model, which is built on a set of basic assumptions that may be listed as follows: 1) there is a wide, well specified, and invariable set of technical methods for producing separately or jointly the various commodities; 2) there are constant returns to scale in the employment of all inputs of production (labour included, in the sense that labour is imputed a subsistence wage rate and is treated — like any other commodity — as an output which requires fixed coefficient inputs of subsistence consumption); 3) the excess of the output of each commodity over the input of the same commodity in the production process is accumulated. Given these assumptions, VON NEUMANN shows that there exists a certain set of techniques and of corresponding prices, and a certain proportion in which the various commodities may be produced, at which a uniform rate of growth of all products is maximum. At this rate of growth, which is considered to be the optimum one, the system grows uniformly in all its sectors, i.e., it multiplies all its sections in the same proportion and therefore keeps constant in time the structure of prices and the relative composition of production.

Professor LEONTIEF arrives at results which are very similar to these, although he does not go into the problem of choice of techniques and starts instead by immediately assuming given production coefficients for each process. LEONTIEF begins with his input-output flow matrix and adds to it a matrix of capital coefficients. Then he shows that there is a well defined proportion among the initial stocks of capital (determined exclusively by the structural coefficients) which yields a maximum uniform rate of growth for all sectors. Even if the system does not start from this particular composition of the initial stocks of capital, it will all the same tend to produce it eventually, although in the meantime it may run into various difficulties.

These have undoubtedly been important steps in the development of dynamic economic analysis. There is in fact nothing in them, which is incompatible with the dynamic model developed in the present work. We may say that they represent a concentration of powerful analytical tools on one particular case: the case of constant returns to scale and no technical progress. Unfortunately, this particular case, elegant and exciting though it may be from a purely analytical point of view, has, on empirical ground, very little practical relevance.

In the foregoing discussion of the relation between the input-output static model and our vertically-integrated dynamic model, there is a point to which all the von Neumann types of models can be traced back. It has been said in section 3 that when, from an analysis of a system at a given point of time, we pass over to considering movements through time, the inverse matrix of the technical coefficients, which provides the analytical bridge between the two models, breaks down, because of technical change. All the von Neumann types of models have been an attempt to resist — for analytical purposes — this hard fact; and to maintain that analytical bridge through time by assumption, if nothing else. But such an assumption, convenient though it is mathematically, has clearly nothing to do with the real world. It means omitting deliberately what has been singled out, in our previous analysis, as the basic force responsible for the dynamism of a modern society, namely the process of learning which goes on, both on the technical and the demand side. In point of fact, one may even question the type of dynamics these models have adopted, which has meant introducing time into a static framework with the careful preoccupation of not affecting the static framework itself. In a sense, time has no importance in these models, since the features of the economic system are described independently of time, once and for all time. As a result, these models do not say about the structure of an economic system anything more than what is already said by the corresponding static model. They add to it a projection through time, by taking the structure of the system at a given point of time and crystallising it — so to speak — for all eternity. The picture which emerges is that of a hypothetical economic system growing only in size but with no development. Each member of the community goes on indefinitely producing the same commodities, quantitatively and qualitatively, receiving the same *per-capita* income and consuming the same consumption goods. We have already discussed this type of system, and acknowledged its logical consistency and beauty, as a mathematical exercise, in chapter III, at the same time pointing out, with reference to any progressive economic system, its lack of practical relevance.

I should not, therefore, come back to the subject if it were not for the fact that once concepts are coined, they tend to be generally used. And although neither VON NEUMANN nor LEONTIEF ever extended their conclusions outside the fixed framework they adopted, the economists who are now-a-days using these concepts do not always appear to be so strict. It is indeed not infrequent, among economists, especially to talk of VON NEUMANN's maximum uniform rate of growth as if it were a concept of general validity; and in particular as if it were applicable to a growing economic system in which there is technical progress. Since extensions of this type are unjustified, and since they distort the very purposes of VON NEUMANN's theoretical scheme, it may be useful to show explicitely why they must be firmly resisted.

Suppose the simplest type of technical progress, from an analytical standpoint; i.e. suppose that improvements take the form of increases in productivity uniformly spread over all sectors. In this case, it is only too natural to abandon

VON NEUMANN's assumption of a subsistence wage rate, and therefore of constant coefficients at which workers can reproduce themselves, and to replace it with the assumption that the wage rate increases in time *pari passu* with productivity. Now we may ask the question: Is it possible to define, at each point of time, a *maximum* technically possible and uniform rate of growth in the VON NEUMANN sense? The answer is yes. But what is the meaning of this maximum uniform rate of growth? It means that, of all possible compositions of total production which follow a uniform rate of growth, there is one at which this uniform rate of growth is maximum. There is something here to which we do not seem to have paid enough attention. The point could have been made earlier with direct reference to the original VON NEUMANN model but it becomes more striking when technical progress is considered. The VON NEUMANN maximum rate of growth entails a very definite *composition* of production, a composition which comes to be determined entirely on technical ground. It means, for example, that to achieve that maximum rate total production will have to be composed by a very high proportion of those commodities which are easier to produce.

Therefore, unless the members of the community are indifferent to the composition of the basket of goods they consume (which would be an absurd assumption to make), i.e., unless the members of the community do not care about whether the national product is mainly composed, let us say, of bread and butter or of juke-boxes, or of nuclear spear-head rockets, the pattern of growth defined by VON NEUMANN's maximum can by no means be called an optimum pattern of growth.

The argument may perhaps be better developed if we follow three successive steps. First of all, let me make the obvious assertion that the members of any society are interested in producing the type of goods they like best and not necessarily the type of goods that are the easiest to produce. Even if we supposed that consumers' preferences were such as to require a uniform expansion of production of all commodities, there is no reason to expect that the composition of production preferred by consumers should coincide with the composition of production which technology would require in order to achieve the maximum rate of uniform growth. If these two compositions do not coincide — as would always be the case, except by a fluke — the composition according to consumers' preferences might evidently rank much higher, in terms of individual utilities and welfare, than the composition according to the von Neumann criterion. The latter would indeed yield a higher rate of overall growth, but many commodities might remain unwanted.

Secondly, as has been argued in Chapter IV, the consumers who enjoy an increasing *per-capita* income do not want a proportional increase of all the commodities they consume. As soon as their demand for each commodity approaches saturation, they are bound to spend the incrasing income on different goods. This leads us to a stronger conclusion than the one reached above. The von Neumann concept of maximum rate of growth, if applied to a system with technical progress, not only may not, but actually can never, correspond — not even by a fluke — to the optimum pattern of growth, i.e., to the pattern of growth that consumers prefer.

Thirdly and finally, there is also the reverse side of the coin. The von Neumann approach imposes on the system the requirement that all commodities should grow at the same rate. This is an unjustified restriction. There might well be just one commodity in the system which is very difficult to produce, in the sense that its production can only grow very slowly through time. Now, in the von Neumann model, which insists on the requirement that all productions must expand at the same rate, that commodity inevitably slows down the growth of the whole system (4). Empirically this is absurd. If that commodity, for example, happens to be one that consumers do not want to increase, there is no reason why the growth of the whole system should be kept back simply in order to fulfil the unjustified requirement of a uniform rate of growth. In practice, the production of that commodity may quite well be kept constant or even decreased or eliminated altogether if better and cheaper substitutes can be invented. All this means that, following consumers' preferences both with regard to the composition of consumption and to the rates of expansion of each single production, it might quite well be possible to achieve an over-all rate of growth which might not only be better, on utility or welfare grounds, but which might also be higher than von Neumann's maximum uniform rate.

To conclude: there is no ground whatsoever for an extension of the von Neumann concept of maximum rate of uniform growth to an economic system in which there is technical progress. In no case would such a maximum rate of uniform growth produce an optimum pattern of growth in terms of utility and welfare. Moreover it may even turn out to be numerically lower than the over-all growth rate achievable by following the sectoral rates of expansion indicated by consumers' preferences (5).

Now the reader may better understand why the dynamic analysis of the previous pages has been freed since the begin-

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(4) This criticism was already raised by D.G. Champernowne in his *A Note on J. V. Neumann's Article on « A Model of General Equilibrium »*, « The Review of Economic Studies », 1945-46. This *Note* is quite indicative of the preoccupations of economists at that time. Mr. Champernowne does point out the limitations of von Neumann's assumption of constant returns to scale but mainly with reference to the case of decreasing returns to scale.

(5) It may be useful to add here that the criticism developed in the text applies in its entirety to the so-called turn-pike theorem, which was proposed by Dorfman-Samuelson-Solow (*op. cit.*, pp. 330-331) and is now so widely discussed in the economic literature.

ning from the strait-jacket of a fixed structure of inter-industry relations, although it has been kept capable of being expanded in such direction whenever needed, with reference to a specific point in time. The pattern of development has thereby appeared in a much simpler and more natural way. The technical evolution of the system has emerged as determining the pattern of costs and therefore of long-run prices, and the evolution of demand, in response to increases in *per-capita* income, as determining the proportions in which the single commodities must be produced. As a result, the solutions of the system, in the form of time-paths of relative prices and quantities, and the over-all rate of growth have emerged as determined and unique; not from any complicated particular or *ad hoc* requirement, but from the simple, common-sense action of technology on the cost side and of consumers' preferences on the side of demand.

CONTENTS

DISCUSSION

MAHALANOBIS

I am tremendously interested in what I have heard. It is extremely exciting to me, if I have understood the general outlook of the paper. I am not an economist and have a very superficial knowledge of marginal analysis and other classical approaches. I should like to check my impressions by asking some questions to see whether I have understood the paper correctly. I am not quite clear about « equilibrium ». The objective is full employment of labour, capital and, I take it, also of all natural resources. How would natural resources come into the picture? I am not quite clear about the implication of the word « equilibrium » in this connexion. The underdeveloped countries have a problem of growth; the equilibrium approach appears to me to be essentially static, because if any function of time is introduced which in some way can be calculated or in some way estimated, that is, of absorbing the time-dimension so to say, then the approach would still remain static. That is, a mere formal inclusion of time does not make the system dynamic; many so-called dynamic models seem to me to be essentially static. The implication of the word « equilibrium » in this context is not clear. In any case, Dr. Pasinetti asked whether any one was interested in such questions; I could very clearly and emphatically declare that there is one person around this table who is.

[10] *Pasinetti* - pag. 119

PASINETTI

Let me express my thanks, first of all, to Professor MAHALANOBIS for his kind words of appreciation. I shall try to answer briefly the three main questions he has raised.

The definition of « equilibrium » which I have chosen is not a static one. It is a definition, used in macro-dynamic analysis, which simply stands for a situation of full employment and full utilization of capacity through time. As far as natural resources are concerned, for the reasons which I tried to explain, I found that introducing them immediately would have put me into those difficulties which have kept back for so long marginal analysis from tackling the problems of economic growth. I have preferred, therefore, to leave them aside, for the time being, although it is my intention to introduce them later on. Finally, I must say that I am in full agreement with Professor MAHALANOBIS on the assertion that simply introducing time into a static model does not make it necessarily dynamic. This is a point which I have tried to make myself, when for example I have criticized the von Neumann-type of growth models, in which time is introduced simply in order to bring about an expansion of the scale of the system, without altering the structure. In such models, the proportions (or structure) of the economic system are specified at a given point in time, and then kept constant for ever; in other words the structure of the system is independent of time. I have always doubted whether we should call such models *dynamic.* They appear to me only *half-dynamic.*

MAHALANOBIS

I think I have understood, and I am in complete agreement about that very important question of structure, because from the point of view of underdeveloped countries, the main objective — at least in the initial stages and for a fairly long time — is to change the structure of the economy. If I have correctly appreciated the point mentioned by Dr. Pasinetti, namely, the deliberate intervention of the human mind in the form of learning and technology, to which I should also like to add the word « science » and also the word « research » covering both science and technology. It will be convenient to define in what way I am using these words, remembering, that all definitions are partly arbitrary. The object of « science research » (including not only the natural sciences but also the social sciences) is to know nature more adequately; the object of « technological research » is to use scientific knowledge to do something more efficiently, that is, to solve practical problems, or to bring about changes in society in an effective way. Using the words in this sense, « science », « technology » and « research » are most important factors, perhaps the most important factors in economic development. All the natural resources have been available since the beginning of the human civilisation; iron ore and other metals and coal, for millions of years; the significant factor was the intervention of the human mind, learning about nature which I am calling science and then trying to utilize such knowledge for useful purposes which I am calling technology. To bring science, technology and research into economics or econometrics seems to me to be a most exciting way of looking on the modern world. Natural resources would also come in because the type of technology to be developed would be determined by the available resources; so long as we have national boundaries technology would have to be oriented towards exploiting given resources. Also, I take it that the object of science, technology, research and learning would be aimed at continually changing the structure; this would be, I think, the truly dynamic part of the work in sharp contrast to the static approach.

DORFMAN

Professor PASINETTI is opening the window on a new and very significant line of investigation. My main reaction is one of hopeful anticipation, but I guess I have two warnings to utter.

In his anxiety to emphasize the dynamic, ever-changing nature of a production-type economy, Professor PASINETTI does less than justice to many of his predecessors, particularly the members of the neoclassical school ranging from WALRAS and JEVONS to SAMUELSON. They, too, were interested primarily in production rather than exchange, and it is from the nature of production that they deduced their theories of distribution and capital. LEONTIEF, by his own admission, is a lineal descendant of WALRAS, and LEONTIEF is par excellence a theorist of production. To be sure he is not a theorist of the aspect of production that intrigues PASINETTI: the process by which production engenders technological progress. PASINETTI's novelty is not that he is concerned with production, but that he is concerned with the evolution of productive techniques.

In their effort to understand exchange and production, the older theorists simplified life by assuming away changes in productive techniques. In pursuit of his new interest, PASINETTI claims the same privilege for himself: he wishes to simplify his life by ignoring factor scarcities. It is no objection to point out that this is a radical proposal. But it may be more of an objection to remember that PASINETTI wishes to introduce prices into his system and to explain them, and it is very hard to say what the social significance of prices is in the absence of scarcity.

PASINETTI

Professor DORFMAN touches upon a really crucial point when, at the very end of his comment, he asks: What is the social significance of prices in the absence of scarcity?

To answer this question, I tbink we must consider the conclusions of the two types of models which have been confronted. In the simplest marginal scheme for scarce goods, quantities are accepted as given by nature and prices emerge as a sort of indexes of scarcity with respect to consumers' preferences. In the theoretical scheme I am proposing (a theoretical scheme for the long run) relative prices are determined by technology. Demand (i.e. consumers' preferences) then determine the relative quantities to be produced. Prices, therefore, emerge as a sort of indexes of *relative efforts* that society is obliged to put into each single unit of the various commodities. The two interpretations of prices are indeed radically different.

The central part of Professor DORFMAN's comment may now be looked at from this point of view. I am, of course, aware that neoclassical economists did deal with problems of production. But I am raising objections to them when they try *to extend* to produced commodities the interpretation of prices and quantities which has emerged from a model for scarce goods. Of course, I am not raising any objection — on the contrary I am approving — when they come near to the interpretation of prices and quantities for produced commodities which I have hinted at above, as for example MARSHALL himself does in his analysis of the long run, and as indeed LEONTIEF does as well.

KOOPMANS

I agree with Dr. PASINETTI that the phenomenon of learning in production has been insufficiently recognized in economic theory. The same applies to learning in consumers' choice. As regards production, I wish to draw attention to a paper by W. Z. HIRSCH, *Firm Progress Ratios*, « Econometrica », April 1956, in which some attempts to quantify the learning process in production are reported on.

PASINETTI

I thank Professor KOOPMANS for his kind bibliographical suggestion. I consider in fact Mr. HIRSCH's paper, referring to the air-frame industry, as one of the many empirical studies which confirm the necessity of the type of analysis which I am trying to propose.

SCHNEIDER

I can be very brief because much of what I wanted to say has been anticipated by Prof. DORFMAN or by Prof. KOOPMANS. I am convinced that Prof. PASINETTI is on the road to important and interesting results; on the other hand I agree with Prof. DORFMAN that his attack on earlier theories and his survey of historical nature is not only unnecessary in this connection but also partially unjust. But I do not want to go into that because I would need an hour to explain my argument.

You want to construct your model independent from the allocation problem. But growth and allocation belong together; they are inter-dependent things and I have an uneasy feeling that you overlook this interdependence.

PASINETTI

Professor SCHNEIDER is right, of course, in saying that optimum allocation of resources and economic growth may not be independent of each other. I do not think, however, that they are necessarily dependent on each other. For example, I can well conceive of an economic system where allocation of resources is optimum, and yet there is absolutely no economic growth whatever. At the same time, I can well imagine an economic system where economic growth is very fast and yet there is misallocation of existing resources. For myself, I have chosen to investigate economic growth first of all.

ALLAIS

I have not had the time to study the paper sufficiently carefully and I have only one question to ask, though there is some discussion of the point on page 47. When considering a system which is growing with a growing population, it is necessary to take into account the greater scarcity of natural resources to be realistic. My question is how can the Prof. PASINETTI's model be modified so as to take into account the limitation of natural resources when population is growing? From a theoretical point of view such a question may not be very important but from a practical point of view it is very important indeed. Thank you.

PASINETTI

I am not sure whether I agree with Professor ALLAIS on the point he has made. My impression is that the problems of scarcity are theoretically very exciting; and yet in practice have not had that importance which our theories have tended to give them.

The bulk of contemporary economic theory has started from the investigation of the optimum allocation of scarce resources in an absolutely stationary world; and has *then* tried to extend the same concepts to a growing economic system. I am proposing a theoretical model which starts from the opposite end; namely from an economic system in which there is no scarcity but there is learning and thus economic growth. Later on — I am hoping — it may well turn out to be easier to introduce scarce resources into a model for learning and growth than it has been so far to introduce learning and growth into a model for scarce resources.

MAHALANOBIS

It is my conviction that one has to develop a model in which there is no scarcity to begin with and to go on later to introduce scarcity. In other words, the first type of model has a kind of conceptual primacy with respect to the second type of a problem — it is only a conceptual primacy — the second type of problem may at each point of time be more important than the other. The first type of approach may be extremely valuable to developing countries.

Professor FRISCH's model is also of great interest to the developing countries because it takes into consideration, in a very imaginative way, of problems which have not been discussed or do not require consideration in connection with the more advanced countries.

The point I am making is that model making and a good deal of econometrics, a very large part of it, very naturally and very properly had started on the basis of the experience of the advanced countries where already a good deal of structural stability had been reached. It was natural and also proper that such developments should have taken place, because these were useful for the advanced countries.

On the other hand the approach of Professor FRISCH and Dr. PASINETTI is likely to prove extremely valuable to underdeveloped countries to supplement the tools already fashioned, and likely to be fashioned in future, to suit the requirements of advanced countries.

THE ROLE OF CAPITAL IN ECONOMIC DEVELOPMENT

MAURICE ALLAIS
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« Science attempts to establish the conceptual identity of things which when first perceived appeared to be different ».

EMILE MEYERSON: *Du cheminement de la pensée* (On the Paths of Thought).

« The truth of a theory in physics is directly related to the number of valid relationships which can be put on record by it ».

HENRI POINCARÉ: *La valeur de la science* (The Value of Science).

« The aim of any theory in physics is to represent experimental laws; the words *truth* and *certainty* have only one meaning in terms of such a theory; they express the agreement between the conclusions to be drawn from the theory and the relationships noted by observers ».

PIERRE DUHEM: *La théorie physique* (Theory of Physics).

Abstract

A general view is given of the results of my work on Capital theory from 1943, date of publication of my *Treatise of Pure Economic Theory,* until 1963.

The first part presents a general formulation of Capital theory, the second a model illustrating the general theory, the third confronts the model with the empirical data, and the fourth presents different applications. In appendix, the model is studied for a very general case.

The different relations given in Part I between quantities which are not real quantities *express simple accounting identities* valid for any economy. Thus they have a very large range of applicability. The relations between real quantities derive simply from the hypothesis that there exists, over a wide range, a valid index $\bar{\bar{R}}_C$ of real consumed national income evaluated at primary factor cost (services of labor and of natural resources) when the process considered is optimal in the Paretian sense. From this hypothesis it follows that

$$\frac{\delta \bar{\bar{R}}}{\bar{\bar{R}}} = k \frac{\Sigma y \delta Y}{\Sigma y Y}$$

where Y represents the primary inputs, y their prices and k the coefficient of homogeneity of the production function. The different relations in real terms of this paper are derived from this hypothesis (§ 119).

The general model rests on the stronger hypothesis that the elasticity

$$\beta = \frac{\delta \overline{R}}{\overline{R}} \Big/ \frac{\delta Y}{Y}$$

of real consumed national income with respect to the primary inputs (the services of labor and natural resources) which can be imputed to it according to Paretian optimality theory can be regarded as practically constant over a wide range (§ 211). While rather stronger, this is still quite a weak hypothesis.

Thus, the theory enables an expression of the real consumed national income to be calculated as a function of the capital output ratio γ.

It is not possible to summarize this paper in a few pages, but to ease the task of the reader, the main definitions and the essential results are presented below (1).

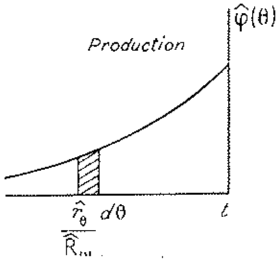
(1) The numbers of the paragraphs and of the formulae are indicated in brackets.

TABLE I — *Main notations and definitions* (§ 110)

	Nominal values	Real values
Reproducible capital	C	$\bar{C}$
National income	R	$\bar{R}$
Consumed national income	R_c	$\bar{R}_c$
Primary income { wages, rents }	R_ω	
Gross national product	P	
Capital-Output ratios	$\gamma = C/R$	$\gamma_c = C/R_c$
Rate of interest		i
Rates of growth — Primary income	$\rho = \frac{1}{R_\omega} \frac{dR_\omega}{dt}$	
Rates of growth — Real income	$\nu = \frac{1}{\bar{R}_c} \frac{d\bar{R}_c}{dt}$	
Rates of growth — Technical progress	$\pi = \frac{1}{\alpha} \frac{d\alpha}{dt}$	

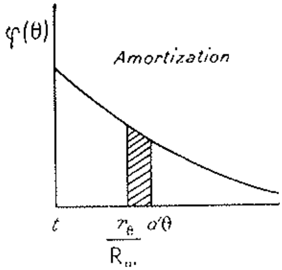
Processes	Quasi stationary processes	Variable rate of growth (§ 120-129)	$\rho = \rho(t)$ $i = i(t)$ $i(t) - \rho(t) = i' =$ constant (120-1) $\varphi(t, 0) = \varphi(0)$ (120-2)
		Constant rate of growth (§ 130-133)	$\rho =$ constant $i =$ constant $\varphi(t, 0) = \varphi(0)$ (120-2) $\hat{\varphi}(t, 0) = \hat{\varphi}(0)$ (130-1)
	Stationary Process (§ 140-142)		$\rho = 0$ $\hat{\varphi}(0) = \varphi(0)$ (140-5) $\hat{R}_\omega = R_\omega$ (140-4)

TABLE II — *Characteristic curves* (§ III)



$$\hat{r}_0 \, d\theta = \hat{R}_\omega \, \hat{\varphi}(\theta) \, d\theta$$

$$r_0 \, d\theta = R_\omega \, \varphi(\theta) \, d\theta$$



$\hat{r}_\theta \, d\,\theta$ = primary input supplied at time $t-\theta$ and emerging in real national consumed income at time t.

$r_\theta \, d\,\theta$ = primary input supplied at time t and emerging in real national consumed income at time $t+\theta$.

$\hat{R}_\omega(t)$ = primary income emerging at instant t.

$R_\omega(t)$ = primary income supplied at instant t.

$$\int_0^\infty \hat{\varphi}(\theta) \, d\theta = 1 \qquad \int_0^\infty \varphi(\theta) \, d\theta = 1 \quad \text{(III-3 and III-4)}$$

Average period of production $\hat{\Theta} = \int_0^\infty \theta \, \hat{\varphi}(t, \theta) \, d\theta$ (II2-1)

Average period of amortization $\Theta = \int_0^\infty \theta \, \varphi(t, \theta) \, d\theta$ (II2-2)

TABLE III — *Main relations*

$$R = R_C + \frac{dC}{dt} = R_\omega + iC \qquad (110\text{-}1)\ (110\text{-}2)$$

$$R_\omega = R_\sigma + R_\varphi \qquad R_\varphi = i\,C_\varphi \qquad (110\text{-}3)\ (110\text{-}4)$$

$$C_T = C + C_\varphi \qquad (110\text{-}5)$$

$$P = R + \hat{R}_\omega \qquad (116\text{-}13)$$

Production function

By hypothesis, real consumed income is a functional of the primary inputs

$$\bar{R}_C(t) = \alpha(t)\,F\left[\hat{R}_\omega(t)\,\hat{\varphi}(t,\theta)\right] \qquad (117\text{-}1)$$

k = coefficient of homogeneity of the production function (§ 117)

Case of a quasi-stationary process with $\rho = \rho\ (t)$

$$R_C(t) = R_\omega(t)\int_0^\infty e^{(i-\rho)\theta}\,\varphi(\theta)\,d\theta \qquad (122\text{-}1)$$

$$\frac{1}{R_C}\frac{dR_C}{dt} = \rho(t) \qquad \frac{1}{C}\frac{dC}{dt} = \rho(t) \qquad (122\text{-}3)\ (123\text{-}2)$$

$$C(t) = \frac{R_C - R_\omega}{i-\rho} \qquad \Theta \leqslant \frac{\Upsilon_C}{1-(i-\rho)\Upsilon_C} \quad \text{for } i \geqslant \rho \qquad (123\text{-}4)\ (123\text{-}12)$$

$$\frac{\delta\bar{R}_C}{\bar{R}_C} = k\,\frac{\int_0^\infty \delta\varphi(\theta)\,e^{(i-\rho)\theta}\,d\theta}{\int_0^\infty \varphi(\theta)\,e^{(i-\rho)\theta}\,d\theta} \qquad (127\text{-}2)$$

$$\bar{R}_C \quad \text{maximum for} \quad i(t) - \rho(t) = 0 \qquad (127\text{-}9)$$

$$\frac{\bar{R}_{CM} - \bar{R}_C}{\bar{R}_{CM}} \sim -k\,\frac{(i-\rho)^2}{2}\,\frac{d\Theta}{d(i-\rho)}\left[i-\rho=0\right] \qquad (127\text{-}10)$$

$\bar{R}_{CM}$ = maximum value of $\bar{R}_C$

All these relations except (117-1) are accounting *identities* [relations (110), (112), (123)] or simple consequences of a Paretian optimality (relations 127).

TABLE IV — *General Model* (Part. II)

HYPOTHESES

(a) *The process is stationary with a variable rate of growth* (§ 210)

$$i\ (t) - \rho\ (t) = i' = \text{constant.}$$

(b) *The production function is logarithmically linear* (§ 211)

$$\frac{d\overline{R}_C}{\overline{R}_C} \bigg/ \frac{d\,\hat{r}_\theta}{\hat{r}_\theta} = \beta(\theta) \qquad (211\text{-}0)$$

$$L\overline{R}_C(t) = L\alpha(t) + \int_0^\infty \beta(\theta)\, L\hat{r}_\theta\, d\theta \qquad (211\text{-}1)$$

[β (θ) of any form]

NOTATIONS (§ 220)

$$\psi(u) = \frac{1}{k}\int_0^\infty \beta(\theta)\, e^{-u\theta}\, d\theta \qquad (220\text{-}1)$$

$$\begin{cases} k = \int_0^\infty \beta(\theta)\, d\theta \qquad \Theta_0 = \dfrac{1}{k}\int_0^\infty \theta\,\beta(\theta)\, d\theta & (211\text{-}3)\ (220\text{-}5) \\ \Delta = \dfrac{1}{2k\Theta_0^2}\int_0^\infty \theta^2 \beta(\theta)\, d\theta & (220\text{-}6) \end{cases}$$

RESULTS

$$\gamma = \frac{C}{R} \sim \Theta_0\,[1 - \Theta_0\, i - \delta\Theta_0\,(i - \rho)] \qquad (240\text{-}11)$$

$$\frac{\overline{R}_C}{\overline{R}_{CM}} = \left[\frac{1 - \gamma\rho}{1 - \gamma i}\, e^{-(i-\rho)\Theta_0}\right]^k \qquad (228\text{-}14)$$

$$\overline{R}_{CM} = K\ (t)\ e^{-k\rho\Theta_0} \qquad \text{for } \rho = \text{constant} \qquad (233\text{-}2)$$

TABLE V — *Exponential model* (§250-253)

HYPOTHESES

(a) and (b) unchanged
(c) constant rate of growth ρ (§ 230)
(d) exponential decrease of the elasticity $\beta(\theta)$ (§ 250)

$$\beta(\theta) = \frac{k}{\Theta_0} e^{-\frac{\theta}{\Theta_0}} \qquad \Delta = 1 \qquad (250\text{-}5) \text{ and } (251\text{-}1)$$

RESULTS

$$\Upsilon = \frac{C}{R} = \widehat{\Theta} = \frac{\Theta_0}{1+\Theta_0 i} \qquad (251\text{-}6) \text{ and } (251\text{-}12)$$

$$C = \Theta_0 R_\omega \qquad (251\text{-}9)$$

$$\frac{\overline{R}_C}{\overline{R}_{CM}} = \left\{ \left[1+\Theta_0(i-\rho)\right] e^{-\Theta_0(i-\rho)} \right\}^k \qquad (251\text{-}15)$$

$$\sim 1 - \frac{k}{2} \Theta_0^2 (i-\rho)^2 \qquad (251\text{-}18)$$

$$\overline{R}_{CM} = \alpha(t) \left[\frac{R_\omega}{e\Theta_0} \right]^k e^{-k\Theta_0\rho} \qquad (251\text{-}17)$$

Θ_0 appears as a quantity whose order of magnitude is practically constant. Θ_0 is an index of the intellectual difficulty of conceiving indirect processes.

TABLE VI — *The general model and the empirical data* (§ 310-319)

I. ESTIMATES OF THE MODEL PARAMETERS

1. *The coefficient of homogeneity* k

$$k \approx 1 \qquad (§\ 311)$$

2. *The function* $\varphi(u)$.

The macroeconomic data give one condition only (§ 312)

$$\gamma = \frac{C}{R} = \frac{1-\psi(i-\rho)}{i-\rho\,\psi(i-\rho)} \sim \Theta_0\left[1-\Theta_0 i-\delta\,\Theta_0(i-\rho)\right] \qquad (312\text{-}1),\ (314\text{-}2)$$

Estimate of Θ_0

$$\Theta_0 \sim \gamma/(1-i\,\gamma) \qquad \gamma = C/R \qquad (§\ 313)$$

Estimate of δ

$$\delta = \Delta - 1 \approx \frac{\gamma_C - \Theta}{\gamma_C^2\,(i-\rho)} \qquad (314\text{-}3)$$

It is impossible to estimate δ with accuracy.

II. EMPIRICAL JUSTIFICATION OF THE GENERAL MODEL

1. *Justification with respect to the hypotheses*
   a) Nature of the production process (Hypothesis *a*) (§ 315).
   b) Constancy of the production elasticities over time (Hypothesis *b*) (§ 316).
   c) Constancy of ρ (Hypothesis *c*) (§ 317).
2. *Justification with the respect to the consequences*
   a) Small variations of $\overline{R}_c$ with $\gamma = C/R$ at a given time (§ 318)

$$\frac{\overline{R}_C(t)}{\overline{R}_{CM}(t)} = \left[\frac{1-\gamma\rho}{1-\gamma i}\, e^{-\Theta_0(i-\rho)}\right]^k \qquad (318\text{-}1)$$

   b) Small variations of $\overline{R}_c$ with $\gamma = C/R$ in time (§ 319).

TABLE VII — *The exponential model and the empirical data* (§ 320 to 325)

I. ESTIMATES OF THE MODEL'S PARAMATERS

Estimates of Θ_0 (§ 321)

$$\Theta_0 = \frac{\Upsilon}{1 - i\Upsilon} \qquad \Upsilon = \frac{C}{R} \qquad (321\text{-}1)$$

Average of Θ_0	United States - Great Britain - France 1913		4.41
	United States	1880-1900	4.34
		1906-1914	4.11
		1950-1956	3.64
		Average 1880-1956	4.12

II. EMPIRICAL JUSTIFICATION OF THE EXPONENTIAL MODEL

2. *Justification with respect to the hypotheses*
   Hypothesis *d*) - Exponential decrease of the production elasticities (§ 322).
2. *Justification with respect to the consequences*
   a) and *b*) small changes of γ at a given time for the various countries and over time for a given country (§323 and 324)

$$\gamma = \Theta_0 / 1 + \Theta_0 i \qquad (323\text{-}1)$$

Estimates of γ					
1913	U.S.A. 3.44	France 3.83	G.B. 3.67	Average 3.64	
U.S.A.	1880-1900 3.56	1906-1913 3.40	1923-1937 3.56	1950-1956 3.35	Average 3.46
World (median of 58 estimates by COLIN CLARK for 21 countries): 3.54					

c) Approximate constancy of C/R_ω at a given time as between countries (§ 325)

$$\frac{C(t)}{R_\omega(t)} = \Theta_0 \qquad \Theta_0 = \frac{\Upsilon}{1 - i\Upsilon} \qquad (325\text{-}1) \text{ and } (325\text{-}3)$$

d) Exponential amortization of primary income (§ 326)

$$\varphi(\theta) = \frac{1}{\Theta} e^{-\frac{\theta}{\Theta}}$$

e) The relation between $\varphi(\theta)$ and the composition of the labor force (§ 330 to 336).

TABLE VIII — *Applications of the model*

1. POSSIBILITY OF INCREASING REAL NATIONAL INCOME BY INCREASING CAPITAL INTENSITY (§ 410-411)

$$g = \frac{\overline{R}_{CM} - \overline{R}_C}{\overline{R}_C} = \frac{\Theta}{\Theta_o} e^{\frac{\Theta_o}{\Theta} - 1} - 1 \qquad (410\text{-}3)\ (410\text{-}4)$$

$$\Theta = \gamma_c = C/R_c \qquad (410\text{-}2)$$

U.S.A. $\gamma_c = 3.35$ $\Theta_o = 4$ $g \sim 1\%$

2. DIMINUTION OF REAL CONSUMED INCOME AS A CONSEQUENCE OF THE INCREASE OF PRIMARY INCOME (§ 420-421)

Decrease of real income per capita

$$p = 1 - e^{-k\Theta_0 \rho} \sim k\Theta_o \rho \qquad (420\text{-}4)$$

U.S.A. $k = 1$ $\Theta_o = 4$ $\rho = 0.017$ $p = 7\%$

3. COMPARISON OF THE PRODUCTIVITY OF TWO COUNTRIES (§ 430-433)

$$\frac{\overline{R}_C^1}{R_\omega^1} \Big/ \frac{\overline{R}_C^2}{R_\omega^2} = e^{-\Theta_0(\rho_1 - \rho_2)} \frac{\Theta_2}{\Theta_1} e^{\Theta_0\left[\frac{1}{\Theta_2} - \frac{1}{\Theta_1}\right]} \qquad (430\text{-}3)$$

Comparison of American and French productivity

	United States/France	
Productivity: ratio	2.3	
Equipment per worker: ratio	2.4	
Output per unit of equipment: ratio	1	
Capital - Output ratio	*U.S.*	*France*
$\gamma = C/R$	3.3 (1956)	$\geqslant$3.3 (1954)

Formula (430-3) shows that it is impossible to explain the greater American productivity by a greater value of Θ, characterizing the capitalistic structure.

4. CAPITALIST DEVELOPMENT POLICY (§ 440).

INTRODUCTION

The purpose of this study is to present a general theory of the role of capital in economic development.

Although there are numerous suggestive practical applications of this theory, the limited space at my disposition obliges me to concentrate mainly on the development of its major theoretical aspects.

This paper follows a long sequence of earlier publications, and it tries to give a general and rigorous presentation of the results obtained earlier.

In chronological order these publications are:

— my *Treatise of Pure Economic Theory* (1943) in which I generalized the theory of optimum allocation of resources for the case of an efficient path over time;

— my book *Economie et Intérêt* (Economy and Interest) (1947), in which are set forth the general principles of the theory of capital which have been the basis of all my subsequent work;

— my study of the influence of the capitalistic structure on the difference between French and United States productivity, published in « Le Monde » in autumn 1948;

— my paper for the 1955 Congress of French language economists, in which the exponential model was presented for the first time;

— my paper for the 1960 Congress of the International Institute of Statistics, held in Tokyo. This study contained an overall assessment of the exponential model together with many numerical applications;

— the analysis of the influence of the capitalistic structure on differences in living standards and productivity as between the French and U.S.A. economies, contained in my 1960 study *L'Europe Unie* (United Europe);

— the Bowley-Walras lecture which I delivered to the Congress of the Econometric Society (Dec. 18th. 1961) at New York, in which I attempted to give an overall view of the work I had done on the theory of capital between 1940 and 1961 (1);

— the monograph which I presented to the Conference of the International Economic Association at Cambridge in July 1963, in which a full demonstration was given of certain propositions which had been stated without proof in the New York lecture (2).

The present study begins with the presentation of a general and condensed formulation of the theory of capital as I have developed it over the past twenty years. This is then illustrated by the discussion of a model of very general application; up to now I have only given summarised versions of this model, or demonstrations of particular aspects of it. I will then show how this model concords with observed data both in respect of its hypotheses and its consequences. Finally, some remarks and certain applications will be given.

It may be that this study is too long, but the reason for this is that I have wanted it to be self-sufficient, providing a complete demonstration of the propositions made, and furnish-

(1) This lecture was published in the October 1962 number of « Econometrica ». However, it was necessary to condense the exposition to such an extent that I fear that this paper is practically unintelligible to those whose who have not had the occasion to become acquainted with my earlier work.

(2) The references to these studies are given in the bibliography.

ing as complete as possible an overall view of the present state of my work on the theory of capital.

To the best of my knowledge, there is no comparable equivalent in current literature, for the approaches adopted elsewhere have been *very different indeed*.

I would like to take this opportunity to add that the results of this study, once established, are of great importance from the standpoint of the orientation of economic development policy. *They show that the capitalistic structure as such has much less importance than is believed by a large segment of opinion.* These results are the culmination of a long sequence of deductions, and it is correspondingly important that each stage in the chain of reasoning be solidly established.

In order to ease the task of the reader, the main notations and formulations have been grouped together in a summary section which precedes this study (1). Some further basic notions which will clarify the approach adopted in the study are given below.

The object of the theory of the capitalistic optimum which is given is, on the one hand, to describe the econometric nature of roundabout methods of production, and also to show that we cannot expect, from an indefinite increase of available real capital, an indefinite increase of real national income consumed per inhabitant. The present study aims to show how the influence of real capital on real income can be measured on the basis of a number of fairly weak assumptions.

The general model which is presented in part II, and its exponential variant in particular, is shown to be very well borne out by all the empirical data which are at present available — confirmation which relates as much to the assumptions as to the results.

(1) Perhaps the best introduction to this study is my 1962 article in « Econometrica »: ALLAIS, *The influence of the Capital-Output Ratio on Real National Income*.

THE ROLE OF CAPITAL IN ECONOMIC DEVELOPMENT

SUMMARY

Part I

GENERAL FORMULATION OF THE THEORY OF CAPITAL

A. — GENERAL CONSIDERATIONS

Definitions

110. The definitions and economic notations to be used are given below [1].

Table I

		Values: in wage units (2)	Values: real
Capital	National reproducible capital [3]	C	$\overline{C}$
	Non-reproducible capital [4] . .	C_φ	
	Total national capital	C_T	
Income [5]	National income	R	$\overline{R}$
	Consumed national income . .	R_C	$\overline{R}_C$
	Primary income [6]	R_ω	
	labour	R_σ	
	natural resources . . .	R_φ	

(1) See Allais (1954), *The accounting basis of macro-economics.*
(2) Nominal values divided by the nominal price of an hour's unskilled labour.
(3) Excluding the capitalised value of natural resources.
(4) The capitalised value of natural resources.
(5) Per unit of time.
(6) The value of primary factors of production (wages and rents) (in French: « revenu originaire »).

(110-1)		$R = R_C + \frac{dC}{dt}$	
(110-2)		$R = R_\omega + i\, C$	
(110-3)	*Relations*	$R_\omega = R_\sigma + R_\varphi$	
(110-4)		$R_\varphi = i\, C_\varphi$	
(110-5)		$C_T = C + C_\varphi$	
(110-6)		$\rho = \frac{1}{R_\omega} \frac{d R_\omega}{dt}$	= Rate of growth of primary income at time t
(110-7)	*Growth rates*	$\nu = \frac{1}{\bar{R}_C} \frac{d \bar{R}_C}{dt}$	= Rate of growth of real consumed national income
(110-8)		π	= Rate of growth of technical progress (1)
		$i(t)$	= Pure rate of interest expressed in wage units
(110-9)	*Capitalistic characteristics*	$\gamma = C/R$	= Capital-output ratio defined in terms of national income
(110-10)		$\gamma_C = C/R_C$	= Capital-output ratio defined in terms of consumed national income.

(1) See below § 117 and 133, and especially the relation (133-2).

The quantity R_ω represents the value of primary factors of production, wages and rents; R stands for National Income, R_C consumed national income, and C the value of reproducible capital. Measurements are made at factor cost. Real values of these variables are indicated by a bar over the symbol concerned.

The equations (110-9) and (110-10) define capital-output as part of reproducible capital and appear in the year's total consumption respectively.

For the sake of generality, consumed national income is defined to include only such consumption as takes place at the time. For example, private motor vehicles are evaluated as part of reproducible capital and appear in the years total consumption only in respect of the services they render.

Characteristic functions

111. This study is based on the concept of the *characteristic function*, which so far as I know was first described by JEVONS in 1871 in the context of a particular model (1). I presented a systematic analysis of this concept in *Economy and Interest* in 1947.

The simplest way of envisaging the curve which is characteristic of a production process (see Figure I below) would appear to be as follows. Assume that in the present period wages are paid in respect of the construction of a blast furnace, which will in due course produce pig-iron. The pig-iron will be used to produce steel, which will in turn find its way into different manufacturing industries, to be used, for example, in making

(1) JEVONS (1871) in chapter VII. The concept of the characteristic function was used for the first time for the case in which $\varphi(0) \neq 0$ for $0 < \theta < \lambda$ and $\varphi(\theta) = 0$ for $\lambda < \theta$.

The concept was subsequently applied in penetrating studies by BÖHM-BAWERK (1888) (see especially Exkurs I to VI) and G.H. BOUSQUET (1936), vol. III, chapter VI.

saucepans which will finally be bought by consumers for their own use.

A part of the wage bill generated in the construction of the blast furnace will finally be incorporated in the price of the saucepan; it will thus appear, with some time lag, in national income.

It can be seen that consumed national income will at a given time include wage outlays which were made at an earlier time, and, at least in theory, it is possible to consider a curve which specifies the distance from the present of various earlier outlays (wages and rents) which appear in the consumed national income of the time t considered. This curve can be called the « *characteristic curve* » of the capitalistic process under study.

If there is a Paretian optimum, factor income is imputed proportionally to the marginal productivities considered in terms of physical values (1).

In parallel with the characteristic curve which defines the origin of the different elements of primary income appearing in the consumed income of time t, an amortisation curve can be defined which represents the time distribution of the primary income of time t over the various subsequent periods (2).

The term $\hat{R}_\omega(t)$ will be used to represent the global amount of primary income per unit of time preceding t which is incorporated in total consumed national income $\overline{R}_c(t)$ of time t. The global amount of primary income per unit of time at time t

(1) This imputation generates a number of very interesting problems which considerations of space preclude me from discussing. On this question, see in particular my *Treatise on Pure Economic Theory* (1943) and my Cambridge paper (1963): *Some Analytical and Practical Aspects of the Theory of Capital.* Appendix I. See also ALLAIS (1961 - C), *The Definition of Characteristic Fonctions and the Problem of Imputation* (La définition des fonctions caractéristiques et le problème de l'imputation).

(2) In general, the definition of these two curves results from the way in which the accounting of imputations is undertaken in practice. If a Paretian optimum obtains, these imputations satisfy the corresponding conditions.

which will be incorporated into national income in periods subsequent to t will be denoted by $R_\omega(t)$.

$\hat{R}_\omega(t)\, \hat{\varphi}(t, \theta)\, d\theta$ is defined as the input primary income between $t-\theta$ and $t-\theta+d\theta$ which appears in the national consumed income of time t, and $R_\omega(t)\, \varphi(t, \theta)\, d\theta$ as the input of primary income of time t incorporated in consumed national income at time $t+\theta$.

From these definition it follows that:

$$\hat{R}_\omega(t) = \int_0^\infty \hat{R}_\omega \hat{\varphi}(t, \theta)\, d\theta \qquad \text{(III-1)}$$

$$R_\omega(t) = \int_0^\infty R_\omega(t)\, \varphi(t, \theta)\, d\theta \qquad \text{(III-2)}$$

These definitions are summarised in Table 2 below

TABLE 2

		Time of input	Appearing in the consumed income of	Corresponding Primary inputs
Primary	Elements	t	$t+\theta$	$R_\omega(t)\varphi(t,\theta)d\theta$
		$t-\theta$	t	$\hat{R}_\omega(t)\hat{\varphi}(t,\theta)d\theta$
income	Global	t		$R_\omega = R_\omega \int_0^\infty \varphi(t,\theta)d\theta$
			t	$\hat{R}_\omega = \hat{R}_\omega \int_0^\infty \hat{\varphi}(t,\theta)d\theta$

Naturally we have

(III-3) $$\int_0^\infty \varphi(t,\theta)\, d\theta = 1$$

(III-4) $$\int_0^\infty \hat{\varphi}(t,\theta)\, d\theta = 1$$

If primary input at instant $(t-\theta)$ but appearing at time t is denoted by $\hat{r}_\theta\, d\theta$, then

(III-5) $$\hat{r}_\theta\, d\theta = \hat{R}_\omega(t)\ \hat{\varphi}\ (t,\ \theta)\ d\theta = R_\omega(t-\theta)\ \varphi\ (t-\theta,\ \theta)\, d\theta$$

whence

(III-6) $$\hat{R}_\omega(t)\ \hat{\varphi}\ (t,\ \theta) = R_\omega(t-\theta)\ \varphi\ (t-\theta,\ \theta)$$

By definition we have

(III-7) $$R_\omega(t) = R_\omega(t-\theta)\, e^{\int_{t-\theta}^{t} \rho(u)\, du}$$

The characteristic curve of primary income appearing at time t is shown in Figure (1).

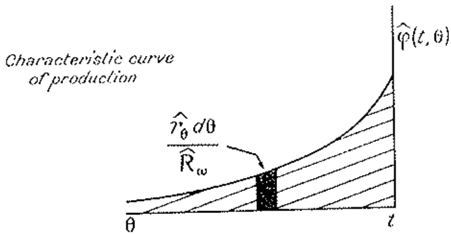


FIG. 1

This curve is obviously asymptotic to the 0 axis, for first there exists no consumable object to day which does not contain some input dating back as far as Julius Caesar ([1]) and second it is clear that the influence of primary inputs tends to zero when the distance in time increases indefinitely.

The characteristic curve for amortisation of primary income furnished at time t but appearing in consumed income at some future moment is given in Figure (2).

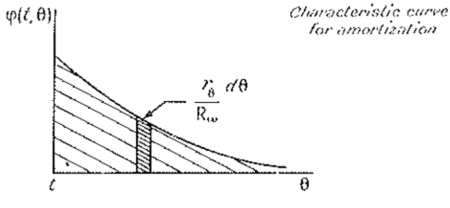


FIG. 2

([1]) Indeed, certain investments made by the Romans are still yielding revenue. A good example is the Coliseum at Rome.

The functions φ (t, θ) *and* $\hat{\varphi}$ (t, θ) *define the capitalistic structure of the economy.*

Average production and amortization periods

112. Taking into account the relations (111-3) and (111-4), an average amortization period Θ and an average production period $\hat{\Theta}$ can be defined by the following formulae:

(112-1) $$\Theta(t) = \int_0^\infty \theta\varphi(t, \theta)\, d\theta$$

(112-2) $$\hat{\Theta}(t) = \int_0^\infty \theta\hat{\varphi}(t, \theta)\, d\theta$$

Income and Capital

113. From the definitions which have been given above, consumed national income (total consumption) at factor cost is given as

(113-1) $$R_C(t) = \int_0^\infty \hat{r}_0\, e^{\int_{t-\theta}^{t} i(u)\, du}\, d\theta$$ (1)

(1) If the global production function with respect to primary inputs is homogeneous of order k and if pricing is proportional to marginal cost

whence

$$(113\text{-}2) \qquad R_c(t) = \hat{R}_\omega(t) \int_0^\infty \hat{\varphi}(t,\theta)\, e^{\int_{t-\theta}^{t} i(u)\,du}\, d\theta$$

whence again from (111-6)

$$(113\text{-}3) \qquad R_c(t) = \int_0^\infty R_\omega(t-\theta)\, \varphi(t-\theta,\theta)\, e^{\int_{t-\theta}^{t} i(u)\,du}\, d\theta$$

so that it follows from (111-7) that

$$(113\text{-}4) \qquad R_c(t) = R_\omega(t) \int_0^\infty \varphi(t-\theta,\theta)\, e^{\int_{t-\theta}^{t} \left[i(u)-\rho(u)\right] du}\, d\theta$$

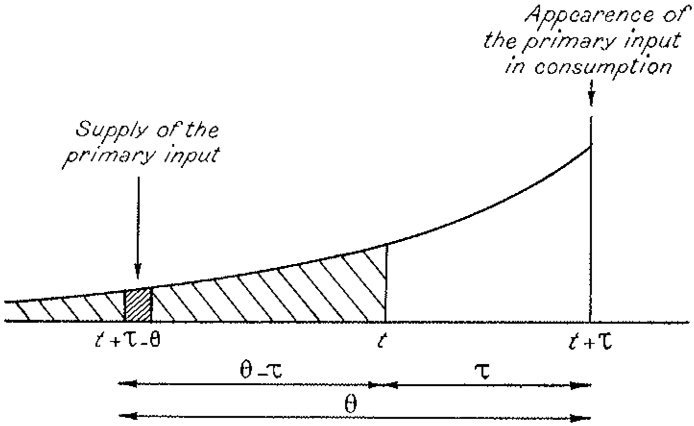
The value C(t) of reproducible capital can be considered as the total capitalized value of the factor inputs furnished prior to instant t which will result in final output only during a period ($t+\tau$, $t+\tau+d\tau$) subsequent to t (1).

(Paretian optimum situation) it can be shown that total consumption valued at market prices will be

$$\frac{1}{k} \int_0^\infty r_\theta\, e^{\int_{t-\theta}^{t} i(u)\,du}\, d\theta$$

(See below, equation (117-12).

(1) ALLAIS (1947), *Economy and Interest*, p. 128.



Input: $\dot{R}_\omega(t+\tau-\theta)\ \varphi\ (t+\tau-\theta,\theta)\ d\theta$

$= \hat{R}_\omega(t+\tau)\ \hat{\varphi}\ (t+\tau,\ \theta)\ d\theta$.

FIG. 3

The value *at instant t* of the primary input furnished prior to t appearing in consumed income $R_C(t+\tau)\,d\tau$ of the period $(t+\tau,\ t+\tau+d\tau)$ is

$$d\tau \int_{\tau}^{\infty} R_\omega(t+\tau-\theta)\ \varphi.(t+\tau-\theta,\theta)\, e^{\int_{t+\tau-\theta}^{t} i(u)\,du}\, d\theta$$

Thus from (111-7)

$$(113\text{-}5) \qquad C(t) = R_\omega(t) \int_0^{+\infty} d\tau \int_\tau^{+\infty} \varphi(t+\tau-\theta,\theta)\, e^{\int_{t+\tau-\theta}^{t} (i-\rho)\,du}\, d\theta$$

Alternatively, changing the order in which the integrations are done, which may be more convenient in certain cases,

$$(113\text{-}6) \quad C(t) = R_\omega(t) \int_0^\infty d\theta \int_0^\theta \varphi(t+\tau-\theta, \theta)\, e^{\int_{t+\tau-\theta}^{t} (i-\rho)\, du}\, d\tau$$

Clearly, it again follows that

$$(113\text{-}7) \quad C(t) = \int_0^\infty d\tau \int_\tau^{+\infty} \widehat{R}_\omega(t+\tau)\, \hat{\varphi}(t+\tau, \theta)\, e^{\int_{t+\tau-\theta}^{t} i(u)\, du}\, d\theta$$

Again, the same result could be derived by integration the differential equation

$$(113\text{-}8) \qquad \frac{d\,C(t)}{dt} - i(t)\, C(t) = R_\omega(t) - R_c(t)$$

by the usual methods and taking into account (113-2), (113-4) and (111-6). In this form, the differential equation is derived from the relations (110-1) and (110-2).

The Value of the Capital which corresponds to the Share of Factor Input which will appear in Consumed Income between Years θ_1 and θ_2:

114. For certain numerical applications, it is useful to evaluate the share $C_{\theta_1}^{\theta_2}(t)$ of capital which corresponds to factor inputs which are frozen in the form of capital over an interval of time included between years θ_1 and θ_2.

We have

(114-1)

$$C_{\theta_1}^{\theta_2}(t) = \iint_S R_\omega(t+\tau-\theta)\,\varphi(t+\tau-\theta,\theta)\, e^{\int_{t+\tau-\theta}^{t} i(u)\,du}\, d\tau\, d\theta$$

where the surface S is represented schematically as below

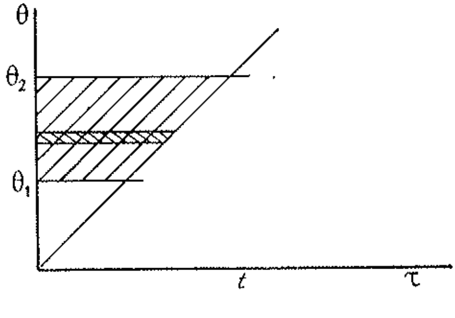


FIG. 4

[11] *Allais* - pag. 28

It is then possible to write

$$(114\text{-}2)\quad C_{\theta_1}^{\theta_2}(t)=\int_{\theta_1}^{\theta_2} d\theta \int_0^{\theta} R_\omega(t+\tau-\theta)\,\varphi(t+\tau-\theta,\theta)\,e^{\int_{t+\tau-\theta}^{t} i(u)du}\,d\tau$$

which gives

$$(114\text{-}3)\quad C_{\theta_1}^{\theta_2}(t)=R_\omega(t)\int_{\theta_1}^{\theta_2} d\theta \int_0^{\theta} \varphi(t+\tau-\theta,0)\,e^{\int_{t+\tau-\theta}^{t}(i-\rho)du}\,d\tau$$

Primary Capital

115. The term « primary capital » $C_\omega(t)$ [1] can be applied to the global value of the factor inputs incorporated in capital $C(t)$. From this definition we have

$$(115\text{-}1)\qquad C_\omega(t)=\int_0^{\infty} d\tau \int_\tau^{+\infty} R_\omega(t+\tau-\theta)\,\varphi(t+\tau-\theta,\theta)\,d\theta$$

(1) In french: « Capital originaire ».

and

$$(115\text{-}2) \qquad C_{\omega}(t) = \int_{0}^{\infty} d\tau \int_{\tau}^{\infty} \hat{R}_{\omega}(t+\tau)\, \hat{\varphi}(t+\tau, \theta)\, d\theta$$

(115-1) and (115-2) can be obtained by putting $i = 0$ in (113-5) and (113-7).

Clearly, using the same notation as in § 114

$$(115\text{-}3) \qquad C_{\theta_1,\omega}^{\theta_2} = R_{\omega}(t) \int_{\theta_1}^{\theta_2} d\theta \int_{0}^{\theta} \varphi(t+\tau-\theta)\, e^{-\int_{t+\tau-\theta}^{t} \rho\, du}\, d\tau$$

This relation is obtained by putting $i = 0$ in equation (114-3).

Macro Economic Magnitudes as Usually Defined and in the Model

116. It seems to me necessary to emphasise the differences between the definitions adopted in the present study and those usually applied to the various economic magnitudes.

To facilitate the exposition, I will denote concepts based on the usual definitions by adding a prime superscript to the appropriate symbols. This will distinguish them from the symbols which refer to magnitudes defined as in the present study, for which the same notation will be used, *but without primes*.

a) *National Capital C*

National capital is equal to the global value C of all reproducible goods, whether production goods (factory buildings, equipment, semi-finished products, stocks), or durable consumer goods (dwelling houses, durable household goods, motor vehicles, etc.).

This definition is of course the definition which is normally used by statisticians in work on the estimation of national wealth, so that

$$\text{(116-1)} \qquad C \sim C' \qquad (^1)$$

b) *Consumed National Income*

In the present theory, consumed national income is equal to the global value of all final services. If, as is of course necessary, consumer durables are included in capital C, the value of the services rendered by them, r, must be included in the calculation of R_C. The income arising from a capital goods is given by

$$\text{(116-2)} \qquad r(t) = a(t) + i(t)\, v(t)$$

where a is amortization, v the value of of the good at instant t, and i the instantaneous rate of interest [2]. This relation sig-

[1] Subject to a number of secondary differences which will be discussed later (§ 334, table on page (115).

[2] ALLAIS, *Traité d'Economie Pure* (Treatise on pure economics) (1943), p. 362, relation 2, and *Economy and Interest* (1947), p. 70, relation 2.

nifies that the value of the services rendered by capital goods is the total of the amortization of all capital goods and the interest on their global value.

Thus, in the present theory, the value of a privately owned and used motor car is not counted in income R_C at the moment of sale to its owner. *This value appears in the form of income only in the form of the services rendered by the car over time.* The definitions usually adopted, by contrast, include the value of an automobile in R'_C at the moment of sale.

There is no difficulty in specifying the correction to be made to R'_C to derive R_C, at least as a first approximation.

The adjustment consists of subtracting investment in consumer durables I_C from consumed income R'_C as it is normally calculated, and adding back the amortisation of consumer durables A_C and the interest iC_C chargeable against the capital represented by existing consumer durables C_C (1).

Thus

$$R_C = R'_C - I_C + A_C + iC_C \ .$$

Now, from the definition of amortization

$$A_C = I_C - \frac{d\ C_C}{dt}$$

$$= I_C - \rho_C\ C_C$$

(1) Relation (116-2) above.

where

$$\rho_C = \frac{1}{C_C} \quad \frac{d\,C_C}{dt} \tag{1}$$

ρ_C denotes the rate of increase of the overall value of holdings of durable consumption goods C_C .

It thus follows that

$$R_C = R'_C + (i - \rho_C)\, C_C \ . \tag{116-3}$$

The adjustment $(i - \rho_C)\, C_C$ is rather small in practice. For the United States in 1965, it is possible to derive (2)

$$i - \rho_C \sim 0.85\%$$
$$C_C \sim 163.4$$
$$R_C \sim 263.1$$

Hence

$$\frac{R'_c}{R_c} = 1 - (i - \rho_C)\,\frac{C_c}{R_c}$$

with

$$(i - \rho_C)\,\frac{C_c}{R_c} = 0.0053 \ .$$

(1) *These formulae should be used in all cases to estimate amortization rather than direct recourse to the corresponding statistics in yearbooks which are quite questionable.*

(2) See § 333-6 below.

c) *National Income*

From relation (110-1), national income is given in the model as

$$(116\text{-}4) \qquad R = R_C + \frac{dC}{dt}$$

and, at least in terms of a first approximation, the concepts which are normally used satisfy the relation

$$(116\text{-}5) \qquad R' = R'_C + \frac{dC'}{dt}$$

It follows immediately from (116-1), (116-3), (116-4) and (116-5) that, at least as a first approximation,

$$(116\text{-}6) \qquad R = R' + (i - \rho_C)\, C_C\ .$$

d) *Gross National Product*

Whereas the differences between C, R_C and R on the one hand, and C′, R'_C and R′ on the other, are relatively insignificant, this is not true for the concepts of Gross National Product (G.N.P.) and overall amortization.

Usual Definitions

Under the definitions usually adopted

$$(116\text{-}7) \qquad P' = R' + A'$$

where P′ is written for G.N.P., and A′ represents the amortization relative to the range of durable goods for the economy as a whole, per unit of time (1).

Further, from (116-5) we have

(116-8) $$P' = R'_C + I'_B$$

with

(116-9) $$I'_B = A' + I'_N$$

(116-10) $$I'_N = \frac{dC'}{dt}$$

where I'_B and I'_N denote gross and net investment respectively.

Under the existing convention of national accounting, capital is for all practical purposes defined as goods whose life expectancy exceeds one year, and indeed in practise, as goods whose life expectancy is at least five years, the shortest period over which amortization is calculated.

As a result of this convention, the lining of a Martin oven, which normally lasts a few months, is written off as an operating expense, whereas the structure of the oven is considered as an investment.

With rules of this type, it is generally found that

(116-11) $$A' \approx K\, R'_C$$

in wich K is of the order of 0.1 to 0.2.

(1) ALLAIS (1954), *Les Fondements comptables de la macroéconomique* (The Accounting Basis of Macroeconomics).

The Case of the Model

Although it facilitates computation, this convention allocates a particular role to a specified interval of time, the year, which is both arbitrary from a theoretical standpoint, and involves running a risk of suggesting erroneous ideas.

It is for this reason that in the present study, investment is defined to include any outlay incorporated in a durable good, *whatever its longevity. Under such a definition any output of primary factors which does not appear immediately as consumed income* is considered as an investment. It is thus a simple matter to see the conceptual changes in the notions of G.N.P., gross investment and amortisation when the framework of the present theory is adopted.

If we consider the concept of a continuous characteristic function $\hat{\varphi}$, depreciation $A\,dt$ is equal to the primary income $\hat{R}_\omega dt$ which emerges in the national consumed income $R_C dt$ of instant t and we have

(116-12) $$A = \hat{R}_\omega$$

whence

(116-13) $$\begin{aligned} P &= R + A \\ &= R + \hat{R}_\omega \end{aligned}$$

using P to correspond to the equivalent in the present theory (in which any investment outlay, whatever its durability, is considered as an investment) of the conventional notion of G.N.P. (in which a minimum durability of one, or even of several years is taken as the boundary between operating expenses and investment outlays).

Thus, in this study, G.N.P. is equal to the sum of national income R and of primary income $\hat{R}_{\omega}$.

These results may seem to be somewhat disconcerting at first sight, but it will become clear after some reflection that they are more natural than those which follow from the adoption of the rather arbitrary concepts usually applied in national accounting.

If, using the usual conventions, amortisation turns out to be only of the order of 20% of consumed national income, the reason is once again that standard systems only allow for the amortization of goods whose life expectancy exceeds one year, or even several years, while all the services invested in a short-period process are not brought into the books as such but are considered as fungible goods.

The example of the Martin furnace has been given already, and a multitude of others could be added to it. Thus, outlays corresponding to certain preparatory work in mining undertakings, those corresponding to a wide variety of agricultural activities, or again a great deal of the maintenance work done in various sectors, for example lorry repair, are considered as operating expenses.

Similarly, the work of the grocer's boy who stocks sardine cans on a shelf is an investment, but it is not counted as such. In the same way, a window cleaner's activity represents an investment, and the only reason it is brought in as an operating expense is that this facilitates the accounting.

Under the formulation which I have given, all that which is not immediately consumed is investment.

Thus, the service rendered by a waiter in a restaurant as he nears the client's table is incorporated in the capital value of that semi-finished good, the dish to be served. This incorporated capital will have been amortized when the customer has completed his repast; but in the conventional evaluation of capital it is not considered as an investment, nor is any account taken of it in the usual estimation of G.N.P.

The value of reproducible capital is several times that of national income. Errors arising from the conventional estimating procedure may not be very considerable relative to capital [1], *but they will be significant in relation to the evaluation of depreciation and of G.N.P.*

The value of the grocer's boy's or of the waiter's services should be counted as a production of capital; it is an element of gross investment and therefore of Gross National Product P.

Demonstration of the Equality $A = \hat{R}_\omega$

It is a simple matter, using the preceding formulation, to show the passage from A′ to A. Conventional accounting only considers as capital those goods whose longevity exceeds a certain duration. Thus with the notation of Table 2 (§ III) the corresponding concept of the general model which is presented in this paper is

$$\text{(116-14)} \qquad A^* = \int_{\theta_1}^{+\infty} \hat{R}_\omega(t)\, \hat{\varphi}(t, \theta)\, d\theta \qquad [2]$$

If the usual convention is modified and θ_1 made to tend to zero, then for $\theta_1 = 0$ relation (III-1) gives

$$\text{(116-15)} \qquad A = A^*[\theta_1 = 0] = \hat{R}_\omega \,.$$

(1) See below § 334, Table 8 and 9.

(2) In fact, under the usual concept the investment in a tool is considered as made when this tool is put in service. With such a concept depreciation A′ includes interest.

Nevertheless from an economic point of view, this is only an intermediate stage and the original investment takes place when the primary output (as defined in § III) are supplied.

The Composition of G.N.P.

Denoting by R_Γ the income given by capital we have

$$R_\Gamma = A + iC \tag{116-16}$$
$$= \hat{R}_\omega + iC$$

so that

$$P = R + A = R + \hat{R}_\omega = R_\omega + R_\Gamma \,. \tag{116-17}$$

G.N.P. is thus equal to the sum of factor income and the income of capital.

The results of the preceding discussion are summarised in the following table ([1]):

TABLE 3

Global Economic Magnitudes		
Concepts	Usual Definition	Defined in Model as
Reproducible capital	C'	$C \sim C'$
Durable consumer goods	C'_c	$C_c = C'_c$
Consumed national income	R'_c	$R_c = R'_c + (i - \rho)\, C_c$
National income	$R' = R'_c + \frac{dC'}{dt}$	$R = R_c + \frac{dC}{dt}$ $R = R' + (i - \rho)\, C_c$
Amortization	$A' \sim 0.2\, R'_c$	$A = \int_0^\infty \hat{R}_\omega(t)\, \hat{\varphi}(t,\theta)\, d\theta$ $A = \hat{R}_\omega$
Gross National Product	$P' = R' + A'$	$P = R + A$
Gross Investment	$I'_B = A' + \frac{dC}{dt}$	$I_B = A + \frac{dC}{dt}$
Net Investment	$I'_N = \frac{dC'}{dt}$	$I_N = \frac{dC}{dt}$ $I_N \sim I'_N$

([1]) A detailed discussion of these notions is given in § 333 and 334 below.

Depreciation in the Production Process and Financial Amortization

The quantity $A = \hat{R}_\omega$ represents the depreciation in the production process. In a similar way it is possible to define a financial amortization. In the present theory any primary outlay is incorporated in capital. Thus we can consider the consumed national income as the sum of financial amortization A_F and of capital interest iC and write

$$R_c = A_F + i\,C\,. \tag{116-18}$$

Then from (110-1) and (110-2) we have

$$R_c - i\,C = R_\omega - \frac{dC}{dt} \tag{116-19}$$

and then

$$A_F = R_\omega - \frac{dC}{dt} \tag{116-20}$$

and taking account of (116-12)

$$A - A_F = \frac{dC}{dt} - \left[R_\omega - \hat{R}_\omega\right] \quad [1] \tag{116-21}$$

(1) The value of $R_\omega - \hat{R}_\omega$ is given below for a special case (relation 251-20).

Case where Consumed Income is equal to Capital Income

In the case where consumed income R_c is equal to capital income R_r we have from (116-16)

$$R_c = \hat{R}_\omega + i\,C \tag{116-22}$$

and from (116-19) we have then

$$\frac{dC}{dt} = R_\omega - \hat{R}_\omega \tag{116-23}$$

Thus in this case the whole increase of capital comes from primary income. There is no contribution from the capitalisation of interest on capital.

From (116-21) and (116-23) we have in this case

$$A = A_F\,. \tag{116-24}$$

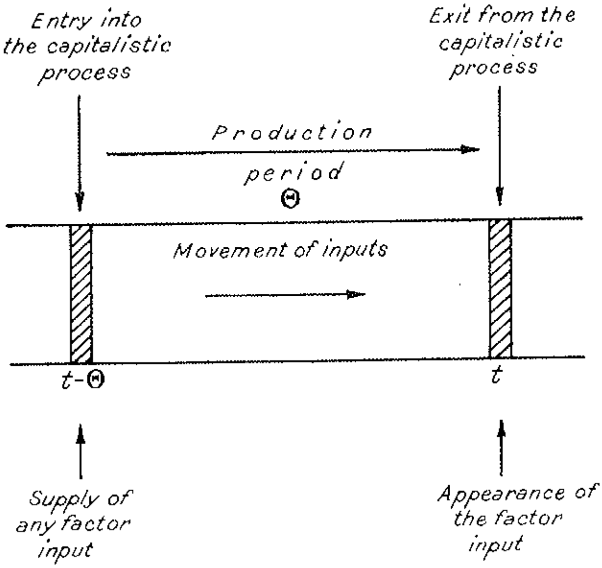
Illustration by a Simple Example

The significance of the preceding discussion can easily be understood by examining a simple example.

Consider a stationary process in which the rate of interest is *zero* and assume that the lag between use of a factor input and its appearance in consumed income R_c is *the same* for all inputs and equal to Θ. Then we have from (116-12), (110-1), (110-2) and § 111.

$$R = R_C = R_\omega = \hat{R}_\omega = A \ . \tag{116-25}$$

The sequence of events in this case can be represented by the following figure, which is the same as that demonstrating the movement of a fluid in a conduit.



This shows that all factor income $R_\omega dt$ arising between $t-\Theta$ and $t-\Theta+dt$ appears between t and $t+dt$ and that in the case of this process it is true that

$$R = R_C = \hat{R}_\omega = R_\omega = A = I_B \ . \tag{116-26}$$

In this particular example all operating expenses would clearly be investment outlays, and a clear cold light would be thrown on the lack of symmetry of the usual definitions.

Real Consumed Income

117. *If it be assumed that real consumed income* $\overline{R}_C$ (t) *is a function of factor inputs* $r_0 d\theta$ i.e. a functional

(117-1)
$$\overline{R}_c(t) = \alpha(t) F\left[\widehat{R}_\omega(t)\,\widehat{\varphi}(t,\theta)\right] = \alpha(t) F\left[R_\omega(t-\theta)\,\varphi(t-\theta,\theta)\right]$$

of the function (111-6)

$$\widehat{R}_\omega(t)\,\widehat{\varphi}(t,\theta) = R_\omega(t-\theta)\,\varphi(t-\theta,\theta)$$

and if it is also assumed that this functional is of k-order homogeneity, then

(117-2)
$$\overline{R}_c(t) = \alpha(t)\left[\widehat{R}_\omega(t)\right]^k G\left[\widehat{\varphi}(t,\theta)\right]$$

In this discussion, $\alpha(t)$ is used as a coefficient which characterises the rate of technical progress.

The characteristic function can in fact be used to derive an expression for $\bar{R}_c$. As the reasoning is somewhat difficult it may be helpful to consider two relatively simple cases before going on to study the general case.

Case 1.

a) Assume in the first place that

$$\bar{R} = \bar{R}\ (A,\ B,\ \ldots,\ C) = \bar{R}(X,\ Y,\ \ldots,\ Z) \tag{117-3}$$

where $\bar{R}$ is real national income, A, B, ..., C represent different outputs (including equipment) and X, Y, ..., Z the factors of production used.

Assuming that $\bar{R}$ (A, B, ..., C) is of first-order homogeneity, and that the function $\bar{R}$ (X, Y, ..., Z) is of k-order homogeneity, then

$$\bar{R}\ (\lambda A,\ \lambda B,\ \ldots,\ \lambda C) = \lambda \bar{R}\ (A,\ B,\ \ldots,\ C) \tag{117-4}$$

$$\bar{R}\ (\mu X,\ \mu Y,\ \ldots,\ \mu Z) = \mu^k \bar{R}\ (X,\ Y,\ \ldots,\ Z)\ . \tag{117-5}$$

Applying the Euler theorem, this gives

$$\Sigma\ B\ \bar{R}'_B = \bar{R} \tag{117-6}$$

$$\Sigma\ Y\ \bar{R}'_Y = k\ \bar{R}\ . \tag{117-7}$$

So that we have

(117-8) $$\frac{\delta \bar{\bar{R}}}{\bar{\bar{R}}} = \frac{\Sigma \bar{\bar{R}}'_B \, \delta B}{\Sigma \bar{\bar{R}}'_B \, B} = k \, \frac{\Sigma \bar{\bar{R}}'_Y \, \delta Y}{\Sigma \bar{\bar{R}}'_Y \, Y}$$

in which the $\delta\bar{\bar{R}}$, δB and δY correspond to virtual displacements compatible with the restraints.

Now, the conditions for Paretian optimum can be written

(117-9) $$\frac{\bar{\bar{R}}'_A}{a} = \frac{\bar{\bar{R}}'_B}{b} = \dots = \frac{\bar{\bar{R}}'_C}{c} = \frac{\bar{\bar{R}}'_X}{x} = \frac{\bar{\bar{R}}'_Y}{y} = \dots = \frac{\bar{\bar{R}}'_Z}{z} = \frac{\Sigma \bar{\bar{R}}'_B B}{\Sigma b B} = \frac{\Sigma \bar{\bar{R}}'_Y Y}{\Sigma y Y}$$

where a, b, ..., c, x, y, ..., z are the prices of the goods A, B ..... C, X, Y ...... Z (1).

Carrying the values of the derivatives of $\bar{\bar{R}}$ given by (117-9) into (117-8), we have

(117-10) $$\frac{\delta \bar{\bar{R}}}{\bar{\bar{R}}} = \frac{\Sigma b \, \delta B}{\Sigma b \, B} = k \, \frac{\Sigma y \, \delta Y}{\Sigma y Y}$$

(1) ALLAIS, *Treatise on Pure Economics* (1943), p. 294.

The first of these relations is usually used to calculate $\delta\ddot{R}$.

In nominal value, the expression for national income is equal to

$$\Sigma\, b\, B$$

at market prices, and to

$$\Sigma\, y\, Y$$

at factor cost. But by (117-9)

$$(117\text{-}11) \qquad \frac{\Sigma\, b\, B}{\Sigma\, y Y} = \frac{\Sigma\, B\, \bar{R}'_B}{\Sigma\, Y\, \bar{R}'_Y}$$

whence it follows from (117-6) and (117-7) that

$$(117\text{-}12) \qquad \Sigma\, b B = \frac{1}{k}\ \Sigma y Y$$

If we consider here only the expression for national income at factor cost, we have

$$(117\text{-}13) \qquad R = \Sigma\, y\, Y$$

and

$$(117\text{-}14) \qquad \frac{\delta \bar{\bar{R}}}{\bar{\bar{R}}} = k \frac{(\delta R)}{R}$$

where (δR) is the differential of national income *calculated at constant prices*.

Case 2.

b) We now consider *a stationary process* ($\rho = 0$) in which R_ω is constant and the functions φ and $\hat{\varphi}$ do not vary over time. The relation (111-6) above shows that the functions φ and $\hat{\varphi}$ are identical [1]. Since the process is stationary, national income and consumed national income are equal. The function $\varphi(\theta)$ which characterises the time distribution of factor inputs is also a function of i, and can be written $\varphi(\theta, i)$.

From (113-4), we have then

$$(117\text{-}15) \qquad R = R_\omega \int_0^\infty \varphi(\theta, i)\, e^{i\theta} d\theta$$

[1] See also § 140 below.

and, using relation (117-14) we deduce

$$\frac{1}{k}\frac{\delta\bar{R}}{\bar{R}} = \frac{\int_0^\infty \delta\varphi\,(\theta, i)\,e^{i\theta}\,d\theta}{\int_0^\infty \varphi\,(\theta, i)\,d\theta} \tag{117-16}$$

In fact, in relation (117-15), the prices of the different factor inputs are equal to the exponentials $e^{i\theta}$. The function $\varphi(\theta)$ characterises the distribution of factor inputs. But, while $\varphi(\theta)$ is of course a function of i it has not been differentiated with i, since in the expression (117-16) the $\delta\varphi$ play an analogous role to the δY in relation (117-10), and since in this relation the y are regarded as constant (1).

Thus the relation (117-16) also enables the expression for the differential of real income to be deduced from the differential of the characteristic function, and we have in particular

$$\frac{1}{k}\,\frac{1}{\bar{R}}\,\frac{\partial\bar{R}}{\partial i} = \frac{\int_0^\infty \frac{\partial\varphi}{\partial i}\,(\theta, i)\,e^{i\theta}\,d\theta}{\int_0^\infty \varphi\,(\theta, i)\,d\theta} \tag{117-17}$$ (2)

(1) Of course this does not mean that i is independent of time. In general i should be considered as a function of time.

(2) These various formulae were first given in 1947 in my study *Economy and Interest* (1947), pp. 186-188.

The General Case of a Dynamic Process

The generalization of these different results follows immediately. In the case of a non-stationary process and assuming k-order homogeneity, we have

$$(117\text{-}18) \qquad \frac{1}{k}\,\frac{\delta \overline{R}_C(t)}{\overline{R}_C(t)} = \frac{\displaystyle\int_0^\infty \delta\hat{\varphi}(t,\theta)\, e^{\int_{t-\theta}^{t} i(u)\,du}\, d\theta}{\displaystyle\int_0^\infty \hat{\varphi}(t,\theta)\, e^{\int_{t-\theta}^{t} i(u)\,du}\, d\theta}$$

where $\delta\overline{R}_C$ represents the change in $\overline{R}_C$ when φ changes by $\delta\varphi$.

Naturally, taking into account equation (111-6) it is further possible to write

$$(117\text{-}19) \qquad \frac{1}{k}\,\frac{\delta \overline{R}_C(t)}{\overline{R}_C(t)} = \frac{\displaystyle\int_0^\infty \delta\left[R_\omega(t-\theta)\,\varphi(t-\theta)\right] e^{\int_{t-\theta}^{t} i(u)\,du}\, d\theta}{\displaystyle\int_0^\infty R_\omega(t-\theta)\,\varphi(t-\theta)\, e^{\int_{t-\theta}^{t} i(u)\,du}\, d\theta}$$

This is the fundamental relation from which the different real values in the present paper are derived.

The Interest Rate and the Rate of Growth

118. I have shown elsewhere that subject to very general conditions a dynamic growth process is characterised asymptotically by the relation

(118-1) $$i(t) = \rho(t) + i'$$

where i is the rate of interest characterising the stationary process which can be made correspond to the process under study (1).

The Significance of the Preceding Relations

119. *All the foregoing relations which do not incorporate real quantities basically express simple accounting identities.* Primary inputs correspond to the accounting imputations which are made.

The relations between the real quantities are simply derived from the assumption of the existence of a satisfactory index of real national income consumed (calculated at factor cost), when the process considered is optimal in the Paretian sense.

When an optimal allocation of resources (2) is realised, the primary inputs and their marginal equalities take on physical significance (3).

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(1) Allais (1963), *Quelques aspects analytiques et appliqués de la théorie du capital* (Some Analytical and Practical Aspects of the Theory of Capital).

(2) In French « Maximation du rendement social ».

(3) See Allais (1963) Cambridge Paper: *Some Analytical and Applied Aspects of the Theory of Capital*, Appendix I.

In the neighborhood of a given situation the fundamental relationship (117-19) is derived from the assumption according to which the relation

$$\frac{\delta \bar{R}_c}{\bar{R}_c} = k \frac{\Sigma y \, \delta Y}{\Sigma y Y} \tag{119-1}$$

is derived from (117-10) in the case of a dynamic process and applies when the use of primary factors (services rendered by labour and natural resources) over time is being studied.

It can thus be seen that the fundamental relationship (117-19) of the present study results simply from the assumption of the existence, over a large range, of a valid index $\bar{R}$ of real national income at factor cost when conditions of optimum allocation of resources in the Paretian sense are satisfied.

B. — A QUASI-STATIONARY PROCESS WITH VARIABLE GROWTH RATE

Definition

120. I will use the expression « a quasi-stationary process with variable growth » to denote a process characterised by

a) invariance of the difference $i(t) - \rho(t)$ over time;

b) invariance of the characteristic function φ over time for a given value of $i - \rho$ (1).

We then have

$$(120\text{-}1) \qquad i(t) - \rho(t) = i'$$

$$(120\text{-}2) \qquad \varphi(t, \theta) = \varphi(\theta)$$

where i' is a constant.

The assumption underlying (120-1) appears to be reasonably acceptable at least as a first approximation. The hypothesis (120-2) is equivalent to assuming that the composition of the labor force is constant over time. This, too, is a quite natural assumption, at least as a first approximation.

These two relations enable the general relationships which have already been derived to be considerably simplified.

(1) The alternative hypothesis in which $\hat{\varphi}(t, 0)$ is assumed invariant over time could also usefully be studied.

Amortization Period

121. We have (relation 112-2)

(121-1) $$\Theta = \int_0^\infty \theta \varphi(\theta)\, d\theta$$

so that

(121-2) $$\frac{d\Theta}{dt} = 0$$

Thus the average duration of the amortization period is invariant over time.

Consumed National Income

122. We deduce from relations (113-4), (120-1) and (120-2)

(122-1) $$R_c(t) = R_\omega(t) \int_0^\infty e^{i'\theta} \varphi(\theta)\, d\theta$$

whence

(122-2) $$\frac{1}{R_c}\,\frac{dR_c}{dt} = \frac{1}{R_\omega}\,\frac{dR_\omega}{dt}$$

so that

$$\frac{1}{R_C}\frac{d R_C}{d t} = \rho(t) \tag{122-3}$$

and finally from (120-1) and (122-1)

$$R_C(t) = R_\omega(t) \quad \text{for} \quad i(t) = \rho(t) . \tag{122-4}$$

Reproducible Capital

123. In the same way from (113-5)

$$C(t) = R_\omega(t)\int_0^\infty d\tau \int_\tau^{+\infty} e^{i'(\theta-\tau)}\varphi(\theta)\, d\theta \tag{123-1}$$

so that

$$\frac{1}{C}\frac{d C}{d t} = \rho(t) \tag{123-2}$$

thus from (122-3) and (123-2)

$$\frac{d}{dt}\gamma_C = \frac{d}{dt}\frac{C(t)}{R_C(t)} = 0 \tag{123-3}$$

Thus the Capital Output ratio defined in terms of consumed national income is constant over time and depends only on i'.

It then follows from relations (110-1) and (110-2) that

$$C(t) = \frac{R_C(t) - R_\omega(t)}{i(t) - \rho(t)} \tag{123-4}$$

This expression gives $C(t)$ without any need for integration by (123-1).

Then from (122-1) we have

$$C(t) = R_\omega(t) \int_0^\infty \frac{e^{i'\theta} - 1}{i'} \varphi(\theta)\, d\theta \tag{123-5}$$

which relation can also be obtained from (123-1) by reversing the order in which the integrations are taken (1).

Again, from (110-1) and (123-2) we have

$$\frac{R_C}{R} = 1 - \gamma \rho \tag{123-6}$$

and from (110-2) and (110-9)

$$\frac{R}{R_\omega} = \frac{1}{1 - \gamma i} \tag{123-7}$$

(1) ALLAIS (1947), « Economy and Interest », pp. 128-130.

From (110-10), (122-1) and (123-5) we have

$$(123\text{-}8)\qquad \int_0^\infty \frac{e^{i'\theta}-1}{i'}\varphi(\theta)\,d\theta = \gamma_c \int_0^\infty e^{i'\theta}\varphi(\theta)\,d\theta$$

Thus developing (123-8) as a Taylor series

$$(123\text{-}9)\qquad \int_0^\infty \left[\theta + i'\frac{\theta^2}{2!} + i'^2\frac{\theta^3}{3!} + \dots\right]\varphi(\theta)d\theta = \gamma_c \int_0^\infty \left[1 + i'\theta + i'^2\frac{\theta^2}{2!} + i'^3\frac{\theta^3}{3!} + \dots\right]\varphi(\theta)\,d\theta$$

Thus from (112-1)

$$(123\text{-}10)\qquad \frac{\gamma_c}{1 - i\gamma_c} = \Theta + \frac{1}{i'}\int_0^\infty \left[i'^2\frac{\theta^2}{2!} + i'^3\frac{\theta^3}{3!} + \dots\right]\varphi(\theta)\,d\theta$$

and finally

$$(123\text{-}11)\qquad \frac{\gamma_c}{1 - i'\gamma_c} = \Theta + \frac{1}{i'}\int_0^\infty (e^{i'\theta} - 1 - i'\theta)\varphi(\theta)d\theta$$

Thus necessarily

$$(123\text{-}12)\qquad \Theta \leqslant \frac{\gamma_c}{1 - (i-\rho)\,\gamma_c} \qquad \text{for} \quad i - \rho \geqslant 0$$

Whatever the function $\varphi(\theta)$ this inequality is very important for empirical analysis.

Again

(123-13) $$\frac{i'^{(n-1)}}{n!}\int_0^\infty \theta^n \varphi(\theta)\, d\theta \longrightarrow 0 \quad \text{for} \quad n=\infty$$

Value of the Capital corresponding to the Share of Factor Income Appearing between Years θ_1 and θ_2

Relation (114-3) can be rewritten

(123-8) $$C_{\theta_1}^{\theta_2}(t) = R_\omega(t)\int_{\theta_1}^{\theta_2} \frac{e^{i'\theta}-1}{i'}\, \varphi(\theta)\, d\theta$$

Clearly, in the light of § 115, and putting $i=0$ in equation (123-8).

(123-9) $$C_{\theta_1,\omega}^{\theta_2}(t) = R_\omega(t)\int_{\theta_1}^{\theta_2} \frac{1-e^{-\rho\theta}}{\rho}\, \varphi(\theta)\, d\theta$$

Primary Capital

From the definition in § 115, we have here

(123-9) $$C_\omega(t) = R_\omega(t)\int_0^\infty \frac{1-e^{-\rho\theta}}{\rho}\, \varphi(\theta)\, d\theta$$

putting $i=0$ and $i' = -\rho$ in relation (123-5) (1).

(1) § 115.

National Income

124. Since

(124-1) $$R = R_c + \frac{dC}{dt}$$

it follows from (123-2) and (123-4) that

(124-2) $$R = R_c + \rho C = R_c + \rho \frac{R_c - R_\omega}{i - \rho}$$

whence finally

(124-3) $$R(t) = \frac{i(t) R_c(t) - \rho(t) R_\omega(t)}{i(t) - \rho(t)}$$

From (123-6) and (123-7) we have

(124-4) $$\frac{R_c(t)}{R_\omega(t)} = \frac{1 - \gamma\rho}{1 - \gamma i} = 1 + \frac{i'}{\frac{1}{\gamma(t)} - i(t)}$$

From § 120 and (122-1) this expression depends only of i', which is evident from (110-2) and (123-5) [or from (125-2) below].

Capital-Output Ratios

125. We have from (110-1)

(125-1) $$R - R_c = \frac{dC}{dt}$$

whence from (110-9), (110-10) and (123-2)

(125-2) $$\frac{1}{\gamma} - \frac{1}{\gamma_c} = \rho\,(t)$$

so that

(125-3) $$\gamma = \frac{\gamma_c}{1 + \rho\,\gamma_c}$$

(125-4) $$\gamma_c = \frac{\gamma}{1 - \rho\,\gamma}$$

and

(125-5) $$(1 + \rho\,\gamma_c)\,(1 - \rho\,\gamma) = 1$$

where γ_c depends only on i' (relation 123.3).

Approximate Expressions for the Principal Magnitudes

126. For small values of i' it is possible to write from (122-1) and (123-5)

$$(126\text{-}1) \qquad R_c(t) = R_\omega(t) \int_0^\infty \left(1 + i'\theta + \frac{i'^2\theta^2}{2} + \dots\right) \varphi(\theta)\, d\theta$$

$$(126\text{-}2) \qquad C(t) = R_\omega(t) \int_0^\infty \frac{i'\theta + \frac{i'^2\theta^2}{2} + \dots}{i'} \varphi(\theta)\, d\theta$$

thus we have from (112-1)

$$\begin{cases} R_c(t) \sim R_\omega(t)\,[1 + i'\,\Theta] & (126\text{-}3) \\ C(t) \sim \Theta\, R_\omega(t) & (126\text{-}4) \end{cases}$$

and according to (110-2)

$$(126\text{-}5) \qquad R(t) \sim R_\omega(t)\,[1 + i\,\Theta]$$

whence

$$(126\text{-}6) \qquad \gamma_c = \frac{C}{R_c} \sim \Theta$$

$$(126\text{-}7) \qquad \gamma = \frac{C}{R} \sim \Theta$$

ignoring first order terms containing i' and ι.

[11] *Allais* - pag. 60

For small values of i' and i we can write

(126)-8) $$R_C(t) \sim R_\omega(t)e^{i'\Theta}$$

(126-9) $$R(t) \sim R_\omega(t)e^{i'\Theta}$$

In terms of a first approximation, the process therefore corresponds to what would happen if consumed national income and national income were equal to their values at the end of a period Θ, applying the respective compounded rates of interest $i' = i - \rho$ and i.

Further from (123-9) and (112-1), for small values of ρ

(126-10) $$C_\omega(t) \backsim \Theta\, R_\omega(t)\ .$$

Real Consumed National Income

127. From (117-19) and for rates of growth $\rho(t)$ considered as given we have

(127-1) $$\frac{1}{k}\,\frac{\delta \overline{R}_C}{\overline{R}_C} = \frac{\int_0^\infty R_\omega(t-\theta)\,\delta\varphi(\theta)\,e^{\int_{t-\theta}^{t} i(u)\,du}\,d\theta}{\int_0^\infty R_\omega(t-\theta)\,\varphi(\theta)\,e^{\int_{t-\theta}^{t} i(u)\,du}\,d\theta}$$

i.e. from (111-7) and (120-1)

(127-2) $$\frac{1}{k}\,\frac{\delta \overline{R}_C}{\overline{R}_C} = \frac{\int_0^\infty \delta\varphi(\theta)\,e^{i'\theta}\,d\theta}{\int_0^\infty \varphi(\theta)\,e^{i'\theta}\,d\theta}$$

or again

$$(127\text{-}3) \qquad \frac{1}{\overline{R}_C} \frac{d\,\overline{R}_C}{di'} = k \frac{\int_0^\infty \frac{d\varphi(\theta)}{di'} e^{i'\theta} d\theta}{\int_0^\infty \varphi(\theta) e^{i'\theta} d\theta}$$

Now

$$e^{i'\theta} = 1 + i'\theta + \frac{i'^2\theta^2}{2} + \dots$$

and from relation (111-3) we have

$$(127\text{-}4) \qquad \int_0^\infty \delta\varphi(\theta)\, d\theta = 0$$

Thus for small values of i' and from (112-1)

$$(127\text{-}5) \qquad \frac{1}{k} \frac{\delta\overline{R}_C}{\overline{R}_C} \sim \frac{i' \int_0^\infty \theta\,\delta\varphi(\theta)\, d\theta}{1 + i'\Theta}$$

whence it follows that

$$(127\text{-}6) \qquad \frac{1}{k} \frac{\delta\overline{R}_C}{\overline{R}_C} \sim \frac{i'\delta i'}{1+i'\Theta} \int_0^\infty \theta \frac{d\varphi(\theta)}{di'}\, d\theta$$

and finally, from (112-1)

(127-7) $$\frac{1}{k}\,\frac{1}{\overline{R}_C}\,\frac{d\overline{R}_C}{di'} \sim \frac{i'}{1+i'\Theta}\,\frac{d\Theta}{di'}$$

When the rates of growth $\rho(t)$ are considered as given, the smaller the $i(t)$ the greater Θ. As from (121-2), Θ is a function of i', we have

(127-8) $$\frac{d\Theta}{di'} < 0$$

It can thus be seen that real consumed income $\overline{R}_C$ is a maximum for $i' = 0$ (1). Thus we have

(127-9) $\overline{R}_C$ is maximum for $i(t) - \rho(t) = 0$.

If we write $\overline{R}_{CM}$ to represent the maximum value of $\overline{R}_C$, we have as a first approximation

(127-10) $$\frac{\overline{R}_{CM} - \overline{R}_C}{\overline{R}_{CM}} \sim k\,\frac{(i-\rho)^2}{2}\,\frac{d\Theta}{d(i-\rho)}\left[i-\rho=0\right]$$

(1) This result was first stated for the case of a dynamic process by DESROUSSEAUX (1961 A), using the formulation I had given in « Economy and Interest » (1947) for a stationary process ($\rho=0$). However, DESROUSSEAUX's line of approach is completely different from that adopted in the present study, for he assumes that the function $\varphi(t, 0)$ satisfies a « condition of regularity » over time, but this condition is formulated from a social, and not an economic, point of view.

The Functions $\hat{R}_\omega(t)$ *and* $\hat{\varphi}(t, \theta)$

128. Lastly, it is possible to deduce from the invariance of φ and from equations (111-6) and (111-7)

$$\hat{R}_\omega(t)\,\hat{\varphi}(t,\theta) = R_\omega(t)\,\varphi(\theta)\,e^{-\int_{t-\theta}^{t}\rho(u)\,du} \tag{128-1}$$

from which it follows that

$$\frac{1}{\hat{R}_\omega}\frac{d\hat{R}_\omega}{dt} + \frac{1}{\hat{\varphi}}\frac{\partial\hat{\varphi}}{\partial t} = \frac{1}{R_\omega}\frac{dR_\omega}{dt} - \left[\rho(t) - \rho(t-\theta)\right] \tag{128-2}$$

Thus from (110-6) we have

$$\frac{1}{\hat{R}_\omega}\frac{d\hat{R}_\omega}{dt} + \frac{1}{\hat{\varphi}}\frac{\partial\hat{\varphi}(t,\theta)}{\partial t} = \rho(t-\theta) \tag{128-3}$$

Nothing further can be said about the functions $\hat{R}_\omega(t)$ and $\hat{\varphi}(t, \theta)$.

Changes in i *and* ρ *over Time*

129. Attention is drawn to the fact that the only assumption implied in the preceding formulas is that the difference $i(t) - \rho(t)$ is constant. They therefore hold where i and ρ are functions of time.

C. — A QUASI-STATIONARY PROCESS WITH CONSTANT GROWTH RATES

Hypothesis of Invariance of the Function $\hat{\varphi}(t, \theta)$

130. *If in addition to the conditions already discussed, the condition that the function* $\hat{\varphi}(t, \theta)$ *is invariant is postulated, it can easily be shown that the rates* i(t) *and* $\rho(t)$ *must be constant.*

For by hypothesis

$$\hat{\varphi}(t, \theta) = \hat{\varphi}(\theta) \tag{130-1}$$

whence, from (128-3)

$$\frac{1}{\hat{R}_\omega} \frac{d\hat{R}_\omega}{dt} = \rho(t-\theta) \tag{130-2}$$

But the left hand member is dependent only on t, and the right hand member only on $t-\theta$. It follows that ρ *must be constant and from* (120-1) *the same is true of the rate of interest* i.

Thus we have

$$\frac{1}{\hat{R}_\omega} \frac{d\hat{R}_\omega}{dt} = \rho$$

A process of this type can be labelled a « quasi stationary process ». It is characterised by the invariance of the rates ρ *and* i *and of the characteristic functions* φ *and* $\hat{\varphi}$.

The fact that i and ρ are constant and the functions φ and $\hat{\varphi}$ invariant enable the formulas obtained for the case of a quasi-stationary process with variable growth rates to be still further simplified.

Characteristic Functions

131. From the relation (112-2) and the fact that $\hat{\varphi}$ is invariant, we deduce that

$$\frac{d\,\widehat{\Theta}}{d\,t} = 0 \qquad (131\text{-}1)$$

i.e. the production period $\hat{\Theta}$ is constant.

Further, from (111-6) we have

$$\hat{\varphi}(\theta)\,\widehat{R}_\omega(t) = R_\omega(t)\,\varphi(\theta)\,e^{-\rho\theta} \qquad (131\text{-}2)$$

and this enables the two conditions below to be deduced from relations (111-3) and (111-4)

$$\widehat{R}_\omega(t) = R_\omega(t)\int_0^\infty e^{-\rho\theta}\,\varphi(\theta)\,d\theta \qquad (131\text{-}3)$$

$$R_\omega(t) = \widehat{R}_\omega(t)\int_0^\infty e^{\rho\theta}\,\hat{\varphi}(\theta)\,d\theta \qquad (131\text{-}4)$$

so that

(131-5) $$\left[\int_0^\infty e^{-\rho\theta}\varphi(\theta)\,d\theta\right]\left[\int_0^\infty e^{\rho\theta}\,\hat{\varphi}(\theta)\,d\theta\right]=1$$

Equation (131-4) implies that $e^{\rho\theta}\hat{\varphi}(\theta)$ tends to zero when θ increases indefinitely.

Equations (131-3), (131-4), (112-1) and (112-2) imply that

(131.6) $$(1-\rho\,\Theta)\,R_\omega(t)\leqslant\hat{R}_\omega(t)\leqslant\frac{R_\omega(t)}{1+\hat{\Theta}\,\rho}\qquad\text{for}\quad\rho\geqslant 0\;.$$

Equations (131-5), (112-1) and (112-2) imply that

(131-7) $$(1-\rho\,\Theta)\,(1+\rho\,\hat{\Theta})\leqslant 1\qquad\text{for}\quad\rho\geqslant 0\;.$$

Thus

(131-8) $$\hat{\Theta}\leqslant\frac{\Theta}{1-\rho\Theta}\qquad\text{for}\quad\rho\geqslant\Theta\;.$$

National Income

132. Since

(132-1) $$R=R_\omega+i\,C$$

and since R_ω and C grow at the rate ρ while i is constant, it follows that

$$(132\text{-}2) \qquad \frac{1}{R}\frac{dR}{dt} = \rho$$

The relationships derived in part IB naturally remain unchanged (§ 120 to 129).

Growth of Real Consumed National Income

133. From (117-2) and (131-3) we deduce

$$(133\text{-}1) \qquad \overline{R}_C(t) = \alpha(t) R_\omega^k(t) \left[\int_0^\infty e^{-\rho\theta} \varphi(\theta)\, d\theta\right]^k G\left[\hat{\varphi}(\theta)\right]$$

In this relation, $\alpha(t)$ represents the impact of technological progress on productivity and we have

$$(133\text{-}2) \qquad \pi(t) = \frac{1}{\alpha(t)}\frac{d\alpha(t)}{dt}$$

where $\pi(t)$ is the rate of growth of technological progress.

As the functions $\varphi(\theta)$ and $\hat{\varphi}(\theta)$ are invariant over time, we have

$$(133\text{-}3) \qquad \frac{1}{\overline{R}_C}\frac{d\overline{R}_C}{dt} = \frac{1}{\alpha(t)}\frac{d\alpha(t)}{dt} + k\frac{1}{R_\omega(t)}\frac{dR_\omega(t)}{dt}$$

whence, from the definitions in § 110,

$$(133\text{-}4) \qquad \nu = \pi + k\,\rho\,.$$

Thus, if the production function is of k-order homogeneity, the rate of growth ν of real consumed national income is given by adding the rate of growth π of technological progress to the product $k\rho$ of the rate of growth of primary income ρ and the coefficient of homogeneity k.

It follows from the preceding that

$$(133\text{-}5) \qquad \pi = \nu - \rho \qquad \text{for} \qquad k = 1\,.$$

In other words, *when the coefficient of homogeneity is equal to unity, which is more or less its value in real conditions, the rate of growth of technical progress is equal to the amount by which the rate of growth of real consumed income exceeds that of primary income. The rate of growth of technical progress then is equal to the rate of growth of productivity per unit of primary income.*

The growth of real consumed income is independent of the capitalistic structure [1], since the functions $\varphi(\theta)$ and $\hat{\varphi}(\theta)$ have been assumed invariant.

[1] In the sense of § 111.

D. — THE CASE OF A STATIONARY PROCESS

General Formulation

140. In the case of a stationary process, we have

(140-1) $$\rho = 0$$

and the preceding formulae simplify considerably.

From relation (131-2) we have then

(140-2) $$\hat{\varphi}(\theta)\hat{R}_\omega = \varphi(\theta)R_\omega$$

so that

(140-3) $$\int_0^\infty \hat{\varphi}(\theta)\,\hat{R}_\omega\,d\theta = \int_0^\infty \varphi(\theta)\,R_\omega\,d\theta$$

whence, from relations (111-3) and (111-4)

(140-4) $$\hat{R}_\omega = R_\omega$$

and therefore

(140-5) $$\hat{\varphi}(\theta)=\varphi(\theta)\ .$$

The relations (122-1), (123-5), (124-3) and (127-3) become

(140-6) $$R=R_C=R_\omega\int_0^\infty e^{i\theta}\varphi(\theta)\,d\theta$$

(140-7) $$C=R_\omega\int_0^\infty \frac{e^{i\theta}-1}{i}\varphi(\theta)\,d\theta$$

(140-8) $$\frac{1}{k}\ \frac{1}{\overline{R}}\ \frac{d\overline{R}}{di}=\frac{\displaystyle\int_0^\infty \frac{d\varphi(\theta)}{di}\,e^{i\theta}\,d\theta}{\displaystyle\int_0^\infty \varphi(\theta)\,e^{i\theta}\,d\theta}$$

and we have as first approximations:

(140-9) $$\gamma=\gamma_c=\frac{C}{R}\sim\Theta$$

(140-10) $$R\sim R_\omega\,(1+i\,\Theta)$$

(140-11) $$\frac{1}{k}\ \frac{\overline{R}_M-\overline{R}}{\overline{R}_M}\sim-\frac{i^2}{2}\ \frac{d\,\Theta}{di}$$

These are the various relations for a stationary process given in my 1947 book « Economy and Interest » (1) (2).

Primary Capital

141. *In a stationary process, the average amortization period is equal to the quotient derived by dividing capital by amortization, whatever the amortization rules.* It follows that for primary capital C_ω we have

$$C_\omega = \Theta\, R_\omega\,. \tag{141-1}$$

From the earlier expression for C (3) and for a given function $\varphi(\theta)$, C_ω can be derived by putting $i=0$. Thus we have

$$\frac{C_\omega}{R_\omega} = \lim_{i=0} \int_0^\infty \frac{e^{i\theta}-1}{i}\,\varphi(\theta)\,d\theta = \int_0^\infty \theta\,\varphi(\theta)\,d\theta \tag{141-2}$$

and this is effectively relation (141-1), (relation 112-1).

Thus Θ *can be regarded as the average amortization period of primary capital* (4).

(1) « Economy and Interest », pp. 127, 128, 132, 133, 187 and 188.
(2) The hypothesis that $k=1$ is made implicitly in this study; the function $\varphi(\theta)$ there considered is the same as $R_\omega\ \varphi(\theta)$ in the notation of the present paper.
(3) Relation (140-7).
(4) Of course this equality also holds as a first approximation for small values of ρ (relation 126-10).

It can further be seen that the interest element incorporated in capital C is

(141-3) $$C - C_\omega = \int_0^\infty \left[\frac{e^{i\theta} - 1}{i} - \theta \right] \varphi(\theta)\, d\theta$$

or, as a first approximation

(141-4) $$C - C_\omega \sim \frac{i}{2} \int_0^\infty \theta^2 \varphi(\theta)\, d\theta$$

Gross National Product

142. In the case of a stationary process, the relations (116-13) and (110-2) can be written from (140-4)

$$P = R + R_\omega = 2R - i\,C\,.$$

Thus, in a stationary process, gross national product is equal to the sum of national income and primary national income.

Since here from (116-12) and (140-4)

$$A = R_\omega$$

it follows that

$$\frac{C_\omega}{A} = \frac{C_\omega}{R_\omega} = \Theta$$

Thus, Θ appears as the average amortization period for primary capital C_ω (§ 141), and since the process considered is a stationary one, this average period Θ is independent of the amortization rule.

For positive rates of interest and from (141-3), C/R_ω is always superior to Θ.

As far as the ratio γ/Θ is concerned and as will be shown later, it may be greater than, smaller than or equal to unity (1), even in the case of a stationary process (2).

(1) § 227, relation (227-6).

(2) § 240 (relations 240-5 and 240-11, with $\rho = 0$).

At the same time, in the case of the exponential model, $\gamma = \Theta$ for $\rho = 0$ (§ 251, relation 251-11).

PART II

MODEL ILLUSTRATING THE GENERAL THEORY

200. The general theory whose main elements have been set forth above can be illustrated by a very simple model. This model has some very suggestive features and appears to be of very wide generality.

Among others, the model has the essential advantage of being easily adapted for numerical applications, and in particular, of facilitating evaluation of the influence of the volume of capital used on real consumed national income.

A. — THE ASSUMPTIONS OF THE MODEL

The model to be examined is based on two hypotheses:

Hypothesis (a): *Paretian Optimum over Time and Invariance of* i(t) – ρ(t)

210. It is assumed that a Paretian optimum over time is realised (1), and that the difference between the rate of growth

(1) In French « maximation du rendement social », or in the terminology of English language literature: « optimum allocation of resources » [see ALLAIS (1943)].

and the rate of interest is a constant, i'.

$$i(t) - \rho(t) = i' \quad (210\text{-}1)$$ [1]

Hypothesis (b): *Constant Production Elasticities*

211. *At any moment* t, *the elasticities* $\beta(\theta)$ *of real consumed income with respect to primary inputs* $\hat{r}_\theta d\theta$ *can be considered as being constant over a wide range and independent of the moment* t *considered.*

As we have

$$\beta(\theta) = \frac{d\,\overline{R}_C}{\overline{R}_C} \bigg/ \frac{d\hat{r}_\theta}{\hat{r}_\theta} \quad (211\text{-}0)$$

we see that, on this hypothesis, real consumed income $\overline{R}_C(t)$ can be represented by the function

$$L\overline{R}_C(t) = L\alpha(t) + \int_0^\infty \beta(\theta) L\hat{r}_\theta \, d\theta \quad (211\text{-}1)$$ [2]

---

(1) These assumptions hold in the case where the process considered respects hypothesis (*c*) in my article in « Econometrica » (ALLAIS, 1962 A) i.e. hypotheses 1-7 of the second version of my paper for the Cambridge Conference (ALLAIS, 1963) *Some Analytical and Practical Aspects of the Theory of Capital.*

(2) The operator L represents the natural logarithm.

with

(2II-2) $$\hat{r}_\theta = R_\omega (t-\theta) \varphi (t-\theta, \theta)$$ (1) (2) (3)

We now write

(2II-3) $$k = \int_0^\infty \beta(\theta) \, d\theta$$

Thus the production function is assumed to be logarithmically linear, and k is assumed to be its degre eof homogeneity. $\alpha(t)$ is a coefficient which depends on t and allows for technical progress, wbich is assumed to operate on a global basis.

---

(1) Relation (111-5) above.
(2) We have

$$\frac{\delta \bar{R}_c}{\bar{R}_c} = \beta(\theta) \frac{\delta \hat{r}_\theta}{\hat{r}_\theta} d\theta$$

Using discrete notation and applying a period analysis of period T, the production function under consideration can be written

(1) $$\bar{R}_c = a \hat{R}_0^{b_0}, \hat{R}_1^{b_1}, \dots, \hat{R}_\theta^{b_\theta}, \dots$$

where a, b_o, b_1, ..., b_θ, ... are constant and $\hat{R}_o$, $\hat{R}_1$, ..., $\hat{R}_\theta$, ..., are the inputs of primary income of moments t, $t-1$, ..., $t-\theta$, ... contributing to real income at moment t.

Thus

(2) $$L\bar{R} = La + b_o L\hat{R}_o + b_1 L\hat{R}_1 , \dots , b_\theta \; L\hat{R}_\theta + \dots$$

B. — CONSEQUENCES OF THE ASSUMPTIONS

Notation

220. It will be useful for what follows to consider the Laplace transform $\varphi(u)$ of the function

(220-1) $$\psi(u) = \frac{1}{k} \int_0^{\infty} \beta(\theta) e^{-u\theta} d\theta$$

---

with

(3) $$\frac{d\overline{R}}{\overline{R}} = b_0 \frac{d\hat{R}_0}{\hat{R}_0}$$

In continuous notation, relation (2) becomes relation (211-1) of the text.
If we put

$$b_0 = T\, \beta(\theta)$$
$$\hat{R}_0 = T\hat{r}_0$$

we have

$$L\overline{R} = La + T\,\beta(0)\,L\,[T\hat{r}_0] + T\,\beta(1)\,L\,[T\hat{r}_1] + \ldots + T\,\beta(\theta)\,L\,[T\hat{r}_\theta] + \ldots$$
$$= [La + \Sigma\, T\,\beta(\theta)\,LT] + \Sigma\, T\,\beta(\theta)\,L\hat{r}_\theta$$

If we suppose that T tends towards zero and that we have

$$\text{limit}\ [La + \Sigma\, T\,\beta(\theta)\,LT] = L\alpha(t)$$

and if we put

$$T = d\,\theta$$

we finally derive

$$L\overline{R} = \alpha(t) + \int_0^{\infty} \beta(\theta)\, L\hat{r}_\theta\, d\theta$$

which is the relation (211-1) of the text.

(3) The meaning of hypothesis (*b*) is in particular that, at least over a large range, there exists a valid index of real national consumed income $\overline{R}_C$. This last property corresponds to the general hypothesis I have made in the first Part (§ 117 and 119) [in particular relations (117-1), (211-1) and 211-2)].

Considering small values of u, developing $\varphi(u)$ as a Taylor series, and taking only the first few terms we have

(220-2) $$\psi(u) \sim 1 - \Theta_0 u + \Delta\Theta^2_0 u^2$$

where Θ_0 and Δ are two constants. We have:

(220-3) $$\frac{d\psi(u)}{du} \sim -\Theta_0 \left[1 - 2\Delta\Theta_0 u\right]$$

(220-4) $$\frac{d^2\psi(u)}{du^2} \sim 2\Delta\Theta_0^2$$

and

(220-5) $$\Theta_0 = \frac{1}{k}\int_0^\infty \theta\beta(\theta)\,d\theta$$

(220-6) $$\Delta = \frac{1}{2k\Theta_0^2}\int_0^\infty \theta^2\beta(\theta)\,d\theta$$

Further

(220-7) $$\psi(0) = 1$$

(220-8) $$\frac{d\psi}{du}(u=0) = -\Theta_0$$

(220-9) $$\frac{d^2\psi}{du^2}(u=0) = 2\Delta\Theta_0^2$$

For a system of points M_i of mass m_i, we have

$$\Sigma m_i \overrightarrow{OM}_i^2 = \overrightarrow{OG}^2 \Sigma m_i + \Sigma m_i \overrightarrow{GM}_i^2 \tag{220-10}$$

where G is the center of gravity. In all cases therefore

$$\Sigma m_i \overrightarrow{OM}_i^2 \geqslant \overrightarrow{OG}^2 \Sigma m_i \tag{220-11}$$

(equality arises only when the entire mass is wholly concentrated in the point G).

We then have by simple transposition

$$\int_0^\infty \theta^2 \beta(\theta) d\theta \geqslant \left[\frac{\int_0^\infty \theta\beta(\theta)\, d\theta}{\int_0^\infty \beta(\theta)\, d\theta}\right]^2 \int_0^\infty \beta(\theta) d\theta \tag{220-12}$$

and from (211-3), (220-5) and (220-6) it follows that of necessity

$$\Delta \geqslant \frac{1}{2} \tag{220-13}$$

Δ will be equal to 1/2 only if the distribution of the $\beta(\theta)d\theta$, *whose overall value is* k, *is concentrated at a given point, and of course this cannot happen in reality* (1).

As will be shown, the value $\Delta = 1$ is a particularly significant one (2).

---

(1) Taking account of (112-1), the same analysis applied to the function $\varphi(\theta)$ shows that

$$\int_0^\infty \theta^2 \varphi(\theta)\, d\theta \geqq \Theta^2 .$$

(2) § 251, relation (251-1).

It is useful to write

(220-14) $$\delta = \Delta - 1$$

with

(220-15) $$\delta \geqslant -\frac{1}{2}$$

Expression for Consumed Income and Real Consumed Income at Factor Cost

221. From relation (113-4) and (210-1) we deduce here

(221-1) $$R_c(t) = R_\omega(t) \int_0^\infty \varphi(t-\theta, \theta) e^{i'\theta} d\theta$$

From (211-1) and (211-2), we have

(221-2) $$L\overline{R}_c(t) = L\alpha(t) + \int_0^\infty \beta(\theta) \left\{ L\left[R_\omega(t-\theta)\varphi(t-\theta,\theta)\right]\right\} d\theta$$

Now,

(221-3) $$R_\omega(t-\theta)=R_\omega(t)e^{-\int_{t-\theta}^{t}\rho(u)\,du}$$

so that

(221-4)

$$L\bar{R}_C(t)=L\alpha(t)+kLR_\omega(t)+\int_0^\infty \beta(\theta)\left\{L\left[e^{-\int_{t-\theta}^{t}\rho(u)du}\varphi(t-\theta,\theta)\right]\right\}d\theta$$

Paretian Optimum Conditions

222. In conditions of Paretian optimum, the cost of production $R_C(t)$ of real consumed income $\bar{R}_C(t)$ is minimized. Subject to the condition of the production function relationship (221-4) at any time t; and according to (221-1) and (221-4) and for values of $\rho(t)$ and $R_\omega(t)$ considered as given, we have

(222-1) $$\int_0^\infty e^{i'\theta}\,\delta\varphi(t-\theta,\theta)\,d\theta=0$$

for any system of values of the $\delta\varphi$ on the condition of restraint

(222-2) $$\frac{\delta \overline{\mathrm{R}}_c}{\overline{\mathrm{R}}_c} = \int_0^\infty \beta(\theta) \frac{\delta \varphi(t-\theta,\theta)}{\varphi(t-\theta,\theta)} \, d\theta = 0$$

where the $\delta\varphi$ represent virtual variations of φ.

This implies that

(222-3) $$e^{i'\theta} = \mathrm{K}(t) \frac{\beta(\theta)}{\varphi(t-\theta,\theta)}$$

from which

(222-4) $$\varphi(t-\theta,0) = \mathrm{K}(t) \beta(\theta) e^{-i'\theta}$$

The Condition $\int_0^\infty \varphi d\theta = 1$

223. From (111-3) we have

(223-1) $$\int_0^\infty \varphi(t,\theta) \, d\theta = 1$$

so that from (222-4)

$$(223\text{-}2) \qquad \int_0^\infty K(t+\theta)\beta(\theta)e^{-i'\theta}d\theta = 1$$

This condition can hold for all values of t *only when* K(t + θ) *is independent of* t + θ, in which case we have

$$(223\text{-}3) \qquad K\int_0^\infty \beta(\theta)e^{-i'\theta}d\theta = 1$$

so that, from (220-1)

$$(223\text{-}4) \qquad K(t) = \frac{1}{k\psi(i')}$$

whence it follows that

$$(223\text{-}5) \qquad \varphi(t-\theta,\theta) = \frac{\beta(\theta)e^{-i'\theta}}{k\psi(i')}$$

As the right hand side is independent of t, it can be seen that φ is independent of t and invariant over time. Thus,

$$(223\text{-}6) \qquad \varphi(t-\theta,\ \theta) = \varphi(\theta)$$

with

$$\varphi(\theta) = \frac{\beta(\theta) e^{-(i-\rho)\theta}}{k\,\psi(i-\rho)} \qquad (223\text{-}7)$$

where i *and* ρ *are functions of time* (§ 210).

It follows that the expression for amortization of factor income is invariant over time and depends only on the difference i − ρ.

The process considered is therefore, following the definition given above, *a quasi stationary process with a variable growth rate,* and the relationships derived under this hypothesis are applicable (1) (2) (3).

---

(1) § 120.

(2) Where the function β depends on time, we have

$$\beta = \beta(t, \theta) \,. \qquad (1)$$

Equation (222-3) can then be written

$$e^{i\theta} = \frac{K(t)\,\beta(t,\theta)}{\varphi(t-\theta,\theta)} \qquad (2)$$

whence

$$\varphi(t, \theta) = K(t+\theta)\,\beta(t+\theta, \theta)\, e^{-i'\theta} \qquad (3)$$

and the integral equation (223-2)

$$\int_0^\infty K(t+\theta)\,\beta(t+\theta, \theta)\, e^{-i'\theta}\, d\theta = 1 \qquad (4)$$

must hold *for all values of* t.

If we write

$$F(t, \theta) = e^{i'\theta}\varphi(t, \theta)$$

then

$$F(t,\theta) = K(t+\theta)\,\beta(t+\theta, \theta)$$

and

$$\int_0^\infty F(t, \theta)\, e^{-i'\theta}\, d\theta = 1 \,.$$

In this case it does not appear that sufficiently simple results can be obtained for them to be easily applied.

(3) It would also be possible to take the invariance of the function φ(θ) as a starting point, deriving the invariance of the function β(θ) as a conclusion. (See ALLAIS, 1960 A, pp. 11 and 12).

The Expression for R_C

224. From (122-1), we derive

(224-1)
$$R_C(t) = R_\omega(t)\int_0^\infty e^{i'\theta}\varphi(\theta)\,d\theta$$

$$= R_\omega(t)\int_0^\infty \frac{\beta(\theta)}{k\,\psi(i')}\,d\theta$$

so that

(224-2)
$$R_C(t) = \frac{R_\omega(t)}{\psi(i-\rho)}$$

National Income

225. From relation (124-3) we have

(225-1)
$$\frac{R}{R_\omega} = \frac{i\dfrac{R_C}{R_\omega} - \rho}{i-\rho}$$

$$= \frac{\dfrac{i}{\psi(i-\rho)} - \rho}{i-\rho}$$

so that

$$\frac{R(t)}{R_c(t)} = \frac{i - \rho\,\psi(i-\rho)}{i-\rho} \tag{225-2}$$

and from (224-2)

$$\frac{R(t)}{R_\omega(t)} = \frac{i-\rho\,\psi(i-\rho)}{(i-\rho)\,\psi(i-\rho)} \tag{225-3}$$

Reproducible Capital and Capital-Output Ratios

226. From (123-4) and (224-2) we have

$$\frac{C}{R_\omega} = \frac{1}{i-\rho}\left[\frac{1}{\psi(i-\rho)} - 1\right] \tag{226-1}$$

from which we derive

$$\gamma_c = \frac{C(t)}{R_c(t)} = \frac{1-\psi(i-\rho)}{i-\rho} \tag{226-2}$$

and

$$\gamma = \frac{C(t)}{R(t)} = \frac{1-\psi(i-\rho)}{i-\rho\,\psi(i-\rho)} \tag{226-3}$$

These relations allow $\psi(i-\rho)$ to be expressed as a function of γ_c and γ, which can be convenient in numerical applications. Thus

$$\psi(i-\rho) = 1 - (i-\rho)\gamma_c \tag{226-4}$$

$$\psi(i-\rho) = \frac{1-\gamma i}{1-\gamma\rho} = 1 - \frac{\gamma(i-\rho)}{1-\gamma\rho} \tag{226-5}$$

Eliminating $\psi(i-\rho)$, we derive

$$\frac{1}{\gamma} - \frac{1}{\gamma_c} = \rho \tag{226-6}$$

Naturally we find again the general relation (125-2) above.

It is further possible to deduce

$$\frac{C(t)}{R_\omega(t)} = \frac{\gamma_c}{1-(i-\rho)\gamma_c} \tag{226-7}$$

$$\frac{C(t)}{R_\omega(t)} = \frac{\gamma}{1-\gamma i} \tag{226-8}$$

and from (224-2), (225-1) and (226-4)

$$\frac{R_c}{R_\omega} = \frac{1}{1-\gamma_c(i-\rho)} \tag{226-9}$$

$$\frac{R}{R_\omega} = \frac{1}{1-\gamma i} \tag{226-10}$$

This last relation can be deduced from (110-2) and (110-9) and has general validity (relation 123-7). The same observation can be made for (226-8) which results from (110-2) and (110-9).

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Primary Capital

From (123-9), (220-1) and (223-7) we have here

$$\frac{C_\omega(t)}{R_\omega(t)} = \frac{1}{\rho}\left[1 - \frac{\psi(i)}{\psi(i-\rho)}\right] \tag{226-11}$$

Average Amortization Period

227. We have from (223-7)

$$\Theta = \int_0^\infty \theta\,\varphi(\theta)\,d\theta = \int_0^\infty \frac{\theta\,\beta(\theta)e^{-i'\theta}}{k\,\psi(i')}\,d\theta \tag{227-1}$$

Now, from (220-1)

$$\frac{d\psi(i')}{di'} = -\frac{1}{k}\int_0^\infty \theta\,\beta(\theta)e^{-i'\theta}d\theta \tag{227-2}$$

so that

$$\Theta = -\frac{1}{\psi(i-\rho)}\,\frac{d\psi}{d(i-\rho)} \tag{227-3}$$

and taking (226-4) into account,

(227-4)
$$\Theta = \frac{1 + \dfrac{i-\rho}{\Upsilon_c} \dfrac{d\Upsilon_c}{d(i-\rho)}}{1-(i-\rho)\Upsilon_c} \Upsilon_c$$

Noting that, according to (226-6) and for $\rho(t)$ *considered* as given, we have

(227-5)
$$\frac{1}{\Upsilon_c^2} \frac{d\Upsilon_c}{d(i-\rho)} = \frac{1}{\Upsilon^2} \frac{d\Upsilon}{d(i-\rho)}$$

we find

(227-6)
$$\Theta = \frac{1}{(1-\Upsilon\rho)(1-\Upsilon i)} \left[1-\Upsilon\rho + \frac{i-\rho}{\Upsilon} \frac{d\Upsilon}{d(i-\rho)}\right] \Upsilon$$

Real Consumed Income

228. From (223-6) and (223-7) we have

(228-1)
$$\mathrm{L}\left[e^{-\int_{t-\theta}^{t} \rho(u)du} \varphi(t-\theta,\theta)\right]$$
$$= -\int_{t-\theta}^{t} \rho(u)du + \mathrm{L}\varphi(\theta)$$
$$= \int_{t-\theta}^{t} \rho(u)du - \mathrm{L}k\psi(i') - i'\theta + \mathrm{L}\beta(\theta)$$

Thus, from equation (221-4) we obtain

(228-2) $$L\overline{R}_C = L\alpha(t) + kL\frac{R_\omega(t)}{k\psi(i')} - i'k\Theta_0 - h(t) + k'$$

with

(228-3) $$h(t) = \int_0^\infty \beta(\theta)\left[\int_{t-\theta}^{t}\rho(u)\,du\right]d\theta$$

(228-4) $$k' = \int_0^\infty \beta(\theta)\,L\beta(\theta)\,d\theta$$

whence finally

(228-5) $$\overline{R}_C(t) = \alpha(t)e^{k'}e^{-h(t)}\left[\frac{R_\omega(t)}{k\psi(i')}\,e^{-i'\Theta_0}\right]^k$$

Maximization of Real Consumed Income for i' = 0

We have

(228-6) $$\frac{1}{k}\frac{\partial}{\partial i'}\left[L\overline{R}_C\right] = -\frac{1}{\psi(i')}\,\frac{d\psi(i')}{d i'} - \Theta_0$$

and

$$(228\text{-}7) \qquad \frac{1}{k}\frac{\partial^2}{\partial i'^2}\left[L\overline{R}_C\right] = \frac{1}{[\psi(i')]^2}\left[\frac{d\psi(i')}{di'}\right]^2 - \frac{1}{\psi(i')}\frac{d^2\psi(i')}{di'^2}$$

Taking relations (220-7), (220-8), (220-9) and (220-13) into consideration, we have

$$(228\text{-}8) \qquad \frac{\partial}{\partial i'}\left[L\overline{R}_C\right] = 0 \qquad \text{for} \quad i = 0$$

$$(228\text{-}9) \qquad \frac{\partial^2}{\partial i'^2}\left[L\overline{R}_C\right] = (1-2\Delta)\Theta_0^2 \leqslant 0 \qquad \text{for} \quad i = 0$$

Thus at any instant, $\overline{R}_C$ is a maximum if

$$(228\text{-}10) \qquad i' = 0 .$$

Therefore we have

$$(228\text{-}11) \qquad \overline{R}_c \ \text{ is a maximum for } \ i(t) = \rho(t) .$$

Writing $\overline{R}_{CM}$ for the maximum value of real consumed national income for $i' = 0$ we have from (228-5)

$$(228\text{-}12) \qquad \frac{\overline{R}_C}{\overline{R}_{CM}} = \left[\frac{e^{-(i-\rho)\Theta_0}}{\psi(i-\rho)}\right]^k$$

with

$$(228\text{-}13) \qquad \overline{R}_{CM} = \alpha(t) e^{k'} \left[\frac{R_\omega(t)}{k} \right]^k e^{-h(t)}$$

where $h(t)$ is a functional of $\rho(t)$ (relation (228-3)) and where k' is given by relation (228-4).

The results presented here are of course particular forms of the general results derived in § 127 above.

Further, taking relations (226-4) and (226-5) into consideration, we have

$$(228\text{-}14) \qquad \frac{\overline{\overline{R}}_C}{\overline{R}_{CM}} = \left[\frac{1}{1-\gamma_C(i-\rho)} e^{-(i-\rho)\Theta_0} \right]^k$$

$$= \left[\frac{1-\gamma_\rho}{1-\gamma i} e^{-(i-\rho)\Theta_0} \right]^k$$

Real National Income

229. From relations (228-5) and (224-2), it is possible to deduce

$$(229\text{-}1) \qquad \overline{R}_C = \frac{\alpha(t) e^{k'-h(t)}}{k^k} e^{-ki'\Theta_0} \left[R_C(t) \right]^k$$

It can thus seen that real consumed income is proportional to $[R_C(t)]^k$.

Then, in order to calculate real national income $\overline{\mathrm{R}}$, it is quite natural to make, *at least as a first approximation, the assumption* that

(229-2)
$$\overline{\mathrm{R}} = \left[\frac{\mathrm{R}}{\mathrm{R_c}}\right]^k \overline{\mathrm{R}}_{\mathrm{c}}$$

which, when $k = 1$, gives

(229-3)
$$\overline{\mathrm{R}} = \frac{\mathrm{R}}{\mathrm{R_c}} \overline{\mathrm{R}}_{\mathrm{c}}$$

This expression can be considered as valid *at least as a first approximation.*

It is then possible to deduce from (124-3) and (229-2)

(229-4)
$$\overline{\mathrm{R}} = \left[\frac{i - \rho \dfrac{\mathrm{R}_\omega}{\mathrm{Rc}}}{i - \rho}\right]^k \overline{\mathrm{R}}_{\mathrm{c}}$$

Whence, bringing in relations (224-2) and (228-12), we derive

(229-5)
$$\frac{\overline{\mathrm{R}}(t)}{\overline{\mathrm{R}}_{\mathrm{CM}}(t)} = \left[\frac{i - \rho\, \psi(i-\rho)}{(i-\rho)\psi(i-\rho)}\; e^{-\Theta_0 (i-\rho)}\right]^k$$

Further, from the relations (226-4) and (226-5) we have

$$\text{(229-6)} \qquad \frac{\bar{\bar{R}}(t)}{\bar{\bar{R}}_{CM}(t)} = \left[\frac{1-\rho\gamma_C}{1+(i-\rho)\gamma_C} \, e^{-\Theta_0 (i-\rho)} \right]^k$$

$$= \left[\frac{1}{1-\gamma i} \, e^{-\Theta_0 (i-\rho)} \right]^k$$

C. — CASE OF A CONSTANT RATE OF GROWTH OF PRIMARY INCOME

Hypothesis (c): *Constant* ρ

230. *It is impossible to obtain simple expressions for* $\hat{\varphi}(t, \theta)$ *and* $\hat{R}_\omega(t)$ *in the more general case where* ρ *is a function of* t, and it does not appear that invariance of ρ, or its equivalent, invariance of the function $\hat{\varphi}(t, \theta)$, can be deduced solely from hypotheses (*a*) and (*b*) (1) (2).

Thus we are led to pose the constancy of ρ *and therefore of* i *as a new hypothesis.*

The Functions $\hat{\varphi}(t, \theta)$ and $\hat{R}_\omega(t)$

231. If the *supplementary* hypothesis is made that the rate of growth $\rho(t)$ is constant, i.e. that the process considered is a quasi-stationary process with a constant rate of growth of primary income, the functions $\hat{\varphi}(t, \theta)$ and $\hat{R}_\omega(t)$ can be specified.

---

(1) § 210 and 211.

(2) Nevertheless it may not be impossible to show that a constant value of ρ is a necessary consequence of the general hypotheses made in my Cambridge paper, from which I deduced the asymptotic property of invariance of $i(t)-\rho(t)$, and hypothesis (*b*). This point still requires further elucidation. Up to now, it has not been possible to make this demonstration.

From relations (128-1) and (223-7)

(231-1) $$\hat{R}_{\omega}(t)\,\hat{\varphi}(t,\theta) = \frac{R_{\omega}(t)\,\beta(\theta)\,e^{-i\theta}}{k\,\psi(i-\rho)}$$

Then from relation (111-4), we have

(231-2) $$\frac{1}{k\psi(i-\rho)}\,\frac{R_{\omega}(t)}{\hat{R}_{\omega}(t)}\int_0^{\infty}\beta(\theta)\,e^{-i\theta} = 1$$

from which we deduce according to (220-1)

(231-3) $$\hat{R}_{\omega}(t) = \frac{\psi(i)}{\psi(i-\rho)}\,R_{\omega}(t)$$

whence, from (231-1)

(231-4) $$\hat{\varphi}(t,\theta) = \hat{\varphi}(\theta)$$

with

(231-5) $$\hat{\varphi}(\theta) = \beta(\theta)\,\frac{e^{-i\theta}}{k\,\psi(i)}$$

The characteristic function $\hat{\varphi}(t, \theta)$ is therefore invariant. Also, we have from (223-7) and (231-5)

(231-6) $$\hat{\varphi}(\theta) = \frac{\psi(i-\rho)}{\psi(i)} e^{-\rho\theta} \varphi(\theta)$$

Since

$$\psi(i - \rho) \geqslant \psi(i) \qquad \text{for} \quad \rho \geqslant 0$$

the function $\hat{\varphi}/\varphi$ declines from a value of above 1 to 0 when θ increases indefinitely.

Average Production Period

232. We have from (112-2) and (231-5)

(232-1) $$\hat{\Theta} = \int_0^\infty \theta \, \hat{\varphi}(\theta) \, d\theta = \int_0^\infty \theta \, \beta(\theta) \frac{e^{-i\theta}}{k\psi(i)} \, d\theta$$

Now, from (220-1), we have

(232-2) $$\frac{d\psi(i)}{di} = -\frac{1}{k} \int_0^\infty \theta \, \beta(\theta) e^{-i\theta} \, d\theta$$

so that

$$\hat{\Theta} = -\frac{1}{\psi(i)} \frac{d\psi(i)}{di} \tag{232-3}$$

From (227-3) and (232-3) we derive

$$\hat{\Theta} \leqslant \Theta \qquad \text{for} \quad \rho \geqslant 0 \tag{232-4}$$

this condition is more restrictive than the condition (131-8) which is valid in a more general case. It could have been derived a priori by observing that for invariant values of the functions $\varphi(\theta)$ and $\hat{\varphi}(\theta)$ over time the effect of growth of R_ω is to reduce $\hat{\Theta}$ in comparison with Θ. It thus follows that Θ seems to be a better indicator of the lengthening of the production process than $\hat{\Theta}$.

It can thus be seen that while the various formulae giving $\hat{R}_\omega$, $\hat{\varphi}(\theta)$ and $\hat{\Theta}$ are dependent on the hypothesis that the rate of growth ρ of primary national income R_ω is constant, the preceding formulae are independent of this assumption.

The Expression for $\overline{R}_{CM}$

233. Relation (228-3) becomes here from (211-3)

$$\begin{aligned} h(t) &= \int_0^\infty \rho\,\theta\,\beta(\theta)\,d\theta \\ &= k\,\rho\,\Theta_0 \end{aligned} \tag{233-1}$$

Thus from (228-13) we have

(233-2) $$\overline{R}_{CM} = K(t)\, e^{-k\rho\Theta_0}$$

with

(233-3) $$K(t) = \alpha(t)\, e^{k'} \left[\frac{R_\omega(t)}{k} \right]^k$$

where $\alpha(t)$ represents the effect of technical progress and k and k' are constants ([1]).

It can thus be seen that at any time, the maximum value of $\overline{R}_{CM}$ which can be attained will be the smaller the greater the value of ρ.

Case of a Stationary Process

234. The formulae corresponding to a stationary process are derived by putting $\rho = 0$ in the preceding relationships. As is to be expected, the general properties of the process given in § 140

(234-1) $$\hat{\varphi} = \varphi, \quad \hat{\Theta} = \Theta, \quad R = R_c, \quad \gamma = \gamma_c, \quad \overline{R} = \overline{R}_c .$$

are again found.

([1]) Relation (211-3) and (228-4).

D. — LIMITED EXPANSION OF THE MAIN EXPRESSIONS FOR SMALL VALUES OF THE RATES ρ AND i

Limited Expansions

240. For numerical applications it is useful to have the limited expansions for the most important of the preceding expressions [1]. Results only are given below, as the calculations offer no difficulty.

General Model

(240-1) $$\psi(i-\rho) \sim 1-\Theta_0(i-\rho)+\Theta_0^2(i-\rho)^2+\delta\Theta_0^2(1-\rho)^2$$

(240-2) $$\hat{R}_\omega \sim (1-\Theta_0\rho)R_\omega$$

(240-3) $$\varphi(\theta) \sim \frac{1}{k}\left[1+\Theta_0(i-\rho)\right]\beta(\theta)\,e^{-(i-\rho)\theta}$$

(240-4) $$\hat{\varphi}(\theta) \sim \frac{1}{k}\left[1+\Theta_0 i\right]\beta(\theta)\,e^{-i\theta}$$

(240-5) $$\Theta \sim \Theta_0\left[1-\Theta_0(i-\rho)-2\delta\Theta_0(i-\rho)\right]$$

(240-6) $$\hat{\Theta} \sim \Theta_0\left[1-\Theta_0 i-2\delta\Theta_0 i\right]$$

(1) Relations (220-14), (220-2), (231-3), (223-7), (231-5), (227-3), (232-3), (224-2), (225-3), (226-1), (226-2), (226-3), (228-12) and (229-2).

$$(240\text{-}7) \qquad R_C/R_\omega \sim 1+\Theta_0(i-\rho)$$

$$(240\text{-}8) \qquad R/R_\omega \sim 1+\Theta_0 i$$

$$(240\text{-}9) \qquad C/R_\omega \sim \Theta_0\left[1-\delta\,\Theta_0\,(i-\rho)\right]$$

$$(240\text{-}10) \qquad \Upsilon_C=C/R_C \sim \Theta_0\left[1-\Theta_0\,(i-\rho)-\delta\,\Theta_0(i-\rho)\right]$$

$$(240\text{-}11) \qquad \Upsilon=C/R \sim \Theta_0\left[1-\Theta_0\,i-\delta\,\Theta_0\,(i-\rho)\right]$$

$$(240\text{-}12) \qquad \frac{\overline{R}_C}{R_{CM}} \sim 1-\frac{k\Theta_0^2}{2}(i-\rho)^2-k\delta\,\Theta_0^2(i-\rho)^2$$

$$(240\text{-}13) \qquad \frac{\overline{R}}{\overline{R}_{CM}} = 1+k\rho\,\Theta_0-\frac{k\Theta_0^2}{2}(i^2-\rho^2)-k\delta\,\Theta_0^2\,i(i-\rho)$$

$$(240\text{-}14) \qquad \frac{\overline{R}_{CM}-\overline{R}_C}{\overline{R}_{CM}} \sim k(1+2\delta)\,\frac{\Theta_0^2}{2}\,(i-\rho)^2$$ (1)

---

(1) This is again the general formula (127-10)

$$\frac{\overline{R}_{CM}-\overline{R}_C}{\overline{R}_{CM}} \sim -k\,\frac{(i-\rho)^2}{2}\,\frac{d\Theta}{d(i-\rho)}\;(i-\rho=0)$$

since from relation (240-5) we have

$$\frac{d\Theta}{d(i-\rho)}\,(i-\rho=0) = -(1+2\,\delta)\,\Theta_0^2$$

This identity of the results is obviously necessary, but it is nonetheless remarkable in that it demonstrates the *underlying consistency of two very different chains of reasoning*; the first starting from (117-19) and the second from (211-1).

These results can be used in numerical applications subject to the assumption that all series expressions converge rapidly (1).

Expressions for the Main Parameters when $i=\rho$ *and* $i=0$

CASE $i=\rho$

241. From the foregoing (2), we have

(241-1) $$\varphi(\theta) = \frac{\beta(\theta)}{k}$$ for $i=\rho$

(241-2) $$\hat{\varphi}(\theta) = \beta(\theta)\frac{e^{-\rho\theta}}{k\psi(\rho)}$$ »

(241-3) $$\Theta = \Theta_0$$ »

(241-4) $$R_c = R_\omega$$ »

(241-5) $$R = [1 + \Theta_0\rho]R_\omega$$ »

(241-6) $$\gamma_C = \frac{C}{R_C} = \Theta_0 = \gamma_{C,0}$$ »

(241-7) $$\gamma = \frac{C}{R} = \frac{\Theta_0}{1+\rho\,\Theta_0} = \gamma_0$$ »

(1) It is possible to check that this is so in practise in a large number of causes, and, as far as can be judged, in all those cases which are of practical significance (See Appendix).

In any case, it is possible for specific cases to confront results based on the development of limit values and those derived from rigorous formulae (in this context, also see the Appendix, § 538).

(2) Relations (223-7), (231-5), (227-3) and (240-5), (224-2), (225-3) and (240-8), (226-2), (226-3) and (220-2).

CASE $i = 0$

242. Few of the formulae take on an interesting form when $i = 0$ and $\rho \neq 0$.

It is only worth noting that

$$\hat{\Theta} = \Theta_o \qquad \text{for } i = 0 \qquad \rho \neq 0 \tag{242-1}$$ (1)

(1) Relations (232-3) and (220-2).

E. — CASE OF AN EXPONENTIAL DECREASE OF ELASTICITY $\beta(\theta)$

Hypothesis (d): *The ratios of the elasticities of primary inputs depends only on their distance in time*

250. If the assumption is made that the ratio of the productivities of two inputs supplied at moments θ_1 and θ_2 ($\theta_2 > \theta_1$) depends only on $\theta_2 - \theta_1$, then considering three moments θ_1, θ_2 and $\theta_3 (\theta_1 < \theta_2 < \theta_3)$ we have from (211-0)

$$\beta(\theta_3 - \theta_1) = \beta(\theta_3 - \theta_2)\,\beta(\theta_2 - \theta_1)\;. \tag{250-1}$$

From the above it follows that

$$\beta(\theta) = \lambda e^{-\mu\theta}\;. \tag{250-2}$$

Then from the relations (211-3) and (220-5)

$$\begin{cases} \mu = \dfrac{1}{\Theta_0} & \text{(250-3)} \\ \lambda = \dfrac{k}{\Theta_0} & \text{(250-4)} \end{cases}$$

whence

$$\beta(\theta) = \frac{k}{\Theta_0} e^{-\frac{\theta}{\Theta_0}} \tag{250-5}$$

Supplementary hypothesis (d) *thus corresponds to an exponential decrease in the elasticities* $\beta(\theta)$.

Characteristics of the Exponential Model

251. We have here from (220-1), (220-6) and (220-14)

$$\psi(u) = \frac{1}{1+\Theta_0 u} \qquad \Delta = 1 \qquad \delta = 0 \tag{251-1}$$

Using this expression, the formulae applying to the exponential model are derived directly from those already given [1]. They are presented in the table below [2].

---

[1] Relations (223-7), (231-5), (227-3), (232-3), (224-2), (225-3), (226-1), (226-9), (226-2), (226-3), (228-12), (229-5), (233-2), (233-3), (251-5) and (251-6).

[2] The notation $\overline{\overline{R}}_M$ represents the maximum values of $\ddot{R}$ for $i = 0$.

Exponential Model

(251-2) $\beta(\theta) = \frac{k}{\Theta_0} e^{-\frac{\theta}{\Theta_0}}$ $\qquad \psi(u) = \frac{1}{1+\Theta_0 u}$

(251-3) $\varphi(\theta) = \frac{1}{\Theta} e^{-\frac{\theta}{\Theta}}$ $\qquad \int_0^\infty \frac{1}{\Theta} e^{-\frac{\theta}{\Theta}} d\theta = e^{-\frac{\theta}{\Theta}}$

(251-4) $\widehat{\varphi}(\theta) = \frac{1}{\widehat{\Theta}} e^{-\frac{\theta}{\widehat{\Theta}}}$ $\qquad \int_0^\infty \frac{1}{\widehat{\Theta}} e^{-\frac{\theta}{\widehat{\Theta}}} d\theta = e^{-\frac{\theta}{\widehat{\Theta}}}$

(251-5) $\Theta = \Theta_0 / [1+\Theta_0(i-\rho)]$ $\qquad i-\rho = \frac{1}{\Theta} - \frac{1}{\Theta_0}$

(251-6) $\widehat{\Theta} = \Theta_0 / [1+\Theta_0 i]$ $\qquad i = \frac{1}{\widehat{\Theta}} - \frac{1}{\Theta_0} \qquad \rho = \frac{1}{\widehat{\Theta}} - \frac{1}{\Theta}$

(251-7) $R_C = [1+\Theta_0(i-\rho)] R_\omega$ $\qquad R_\omega = [1-\Theta(i-\rho)] R_C$

(251-8) $R = [1+\Theta_0 i] R_\omega$ $\qquad \widehat{R}_\omega = \frac{1+\Theta_0(i-\rho)}{1+\Theta_0 i} R_\omega$

(251-9) $C = (1+\Theta_0 i) C_\omega$ $\qquad \boxed{C = \Theta_0 R_\omega}$

(251-10) $C_\omega = \frac{\Theta}{1+\Theta\rho} R_\omega = \frac{\Theta_0}{1+\Theta_0 i} R_\omega = \widehat{\Theta} R_\omega$

(251-11) $\gamma_C = C/R_C = \Theta$ $\qquad \Theta = \frac{\gamma}{1-\rho\gamma}$

(251-12) $\gamma = C/R = \widehat{\Theta} = C_\omega / R_\omega$ $\qquad \frac{1}{\gamma} - \frac{1}{\gamma_C} = \rho(t)$

$$(251\text{-}13)\quad \gamma_{C,0} = \gamma_C\,(i=\rho) = \Theta_0 \qquad\qquad \boxed{\Theta_0 = \frac{\gamma}{1 - i\,\gamma}}$$

$$(251\text{-}14)\quad \gamma_0 = \gamma\,(i=\rho) = \Theta_0 / (1 + \rho\,\Theta_0)$$

$$(251\text{-}15)\quad \frac{R_C}{\overline{\overline{R}}_{CM}} = \left[\frac{\Theta_0}{\Theta}\, e^{1 - \frac{\Theta_0}{\Theta}}\right]^k \qquad \boxed{\frac{\overline{R}_C}{\overline{R}_{CM}} = \left\{\left[1 + (i-\rho)\Theta_0\right] e^{-\Theta_0 (i-\rho)}\right\}^k}$$

$$(251\text{-}16)\quad \frac{\overline{\overline{R}}}{\overline{\overline{R}}_M} = \left[\frac{\Theta_0}{\widehat{\Theta}}\, e^{1 - \frac{\Theta_0}{\widehat{\Theta}}}\right]^k \qquad \boxed{\frac{\overline{R}}{\overline{\overline{R}}_M} = \left[(1 + \Theta_0\, i)\, e^{-\Theta_0\, i}\right]^k}$$

$$(251\text{-}17)\quad \overline{R}_{CM} = \alpha(t)\left[\frac{R_\omega}{e\,\Theta_0}\right]^k e^{-k\,\Theta_0\,\rho} \qquad \overline{R}_M = \alpha(t)\left[\frac{R_\omega}{e\,\Theta_0}\right]^k$$

In these formulae, R_ω, R_C, R, $\overline{R}_C$, $\overline{R}$, i and ρ are functions of time. The quantities $(i-\rho)$, γ_C, γ, Θ, $\widehat{\Theta}$, $\overline{R}_C/\overline{\overline{R}}_{CM}$, $\overline{R}/\overline{\overline{R}}_M$ are invariant.

Those formulae in which $\widehat{\varphi}(\theta)$ and $\widehat{\Theta}$ appear *involve the supplementary hypothesis* that i and ρ are constants (1). In the same way the expressions (251-17) given for $\overline{R}_{CM}$ and $\overline{R}_M$ are valid only on this hypothesis (2).

It can thus be seen that in the exponential hypothesis, the significant expressions depend on only two coefficients: the coefficient of homogeneity k, *and the time constant* Θ_0.

Of all these relations, perhaps the most striking is that which specifies that the ratio C/R_ω is a constant and equal to

(1) § 230 to 232.

(2) Relations (233-2), (233-3), (228-4) and (250-5) for $\overline{R}_{CM}$; relations (229-5), (251-1) and first relation (251-17) for $\overline{R}_M$.

the constant Θ_o. This condition signifies that the value of capital per unit of primary income is independent of the rates of interest and growth i and ρ (1).

Of course it is again found that $\bar{\bar{R}}_c$ *is maximum for* $i - \rho = 0$. But it will also be remarked that national income $\bar{\bar{R}}$ is maximum for $i = 0$ whatever ρ. This latter is not a general finding; it is true only for the exponential model. The notation $\bar{\bar{R}}_M$ represents the maximum value of $\bar{\bar{R}}$.

It should also be observed that relation (251-12) between γ, γ_c and ρ is independent of the model. It is based only on the constancy of $(i - \rho)$ and the invariance of the function $\varphi(\theta)$ over time [§120 and relation (125-2)].

Expansions in series form as functions of i and $i - \rho$ are done as previously (2) by putting

$$\delta = 0$$

which value corresponds to the exponential expression for β.

For small values of $(i - \rho)$ we find here

$$\frac{\bar{\bar{R}}}{\bar{\bar{R}}_{CM}} \sim 1 - \frac{k}{2}\,\Theta_o^2\,(i-\rho)^2 \qquad (251\text{-}18)$$

and for the small values of i

$$\frac{\bar{\bar{R}}}{\bar{\bar{R}}_{M}} \sim 1 - \frac{k}{2}\,\Theta_o^2\,i^2 \qquad (251\text{-}19)$$

(1) § 240.

(2) In my Tokyo monograph (ALLAIS, 1960 A) I derived and made detailed comments on the different relations given above for the case $\rho = 0$.

Value of $R_\omega - \hat{R}_\omega$

In the case of the exponential model, we have, according to (251-7)

$$(251\text{-}20) \qquad R_\omega - \hat{R}_\omega = \left[1 - \frac{1+\Theta_0(i-\rho)}{1+\Theta_0\, i}\right] R_\omega$$

$$\sim \rho\, \Theta_0 R_\omega$$

for small values of i and ρ. Now, from (110-6) and (251-9), we have

$$(251\text{-}21) \qquad \frac{dC}{dt} = \Theta_0 \frac{dR_\omega}{dt}$$

$$= \rho\, \Theta_0 R_\omega$$

Thus we have

$$(251\text{-}22) \qquad R_\omega - \hat{R}_\omega \sim \frac{dC}{dt} \qquad (^1)$$

Thus, in the exponential case consumed income is practically equal to capital income (see § 116) and we have from (116-22), (251-20) and (110-2)

$$(251\text{-}23) \qquad R_c \sim \hat{R}_\omega + i\, C$$

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(1) We have also this property in the general case. In this case, we have from (231-3), (123-2), (123-4), (224-2) and (220-2)

$$R_\omega - \hat{R}_\omega = \left[1 - \frac{\psi(i)}{\psi(i-\rho)}\right] R_\omega \sim \rho\, \Theta_0\, R_\omega$$

$$\frac{dC}{dt} = \rho\, C = \rho \frac{R_c - R_\omega}{i - \rho} = \rho \frac{[1 - \psi(i-\rho)]}{(i-\rho)\,\psi(i-\rho)} R_\omega \sim \rho\, \Theta_0\, R_\omega$$

Here, from (116-13), (110-1), (251-21), (251-7), gross national product as understood in this paper is given as

(251-24)
$$P = R + \hat{R}_\omega$$
$$= R_c + \frac{dC}{dt} + \hat{R}_\omega$$
$$= R_c + \rho\,\Theta_0\,R_\omega + \frac{1+\Theta_0\,(i-\rho)}{1+\Theta_0\,i}\,R_\omega$$
$$= R_c + \left[1 + \frac{\Theta_0^2\,\rho\,i}{1+\Theta_0\,i}\right] R_\omega$$

thus

(251-25)
$$P \sim R_c + R_\omega$$

disregarding terms beyond the second order in i and ρ. It is interesting to note that:

(251-26)
$$\hat{\Theta} = \frac{1+\Theta\rho}{\Theta} \leqslant \Theta \qquad \text{for} \quad \rho \geqslant 0$$

Finally, relations (251-8) and (251-9) show that the share of interest in national income and in national reproducible capital is the same.

Value of the Capital corresponding to the Share of Primary Income which subsequently appears in National Income between years θ_1 and θ_2.

252. From relation (123-8) and writing $i - \rho = u$ we have

$$(252\text{-}1)\quad C_{\theta_1}^{\theta_2}(t) = R_\omega(t)\frac{1}{u\Theta}\left\{\frac{e^{-\left(\frac{1}{\Theta}-u\right)\theta_1} - e^{-\left(\frac{1}{\Theta}-u\right)\theta_2}}{\frac{1}{\Theta}-u} - \Theta\left[e^{-\frac{\theta_1}{\Theta}} - e^{-\frac{\theta_2}{\Theta}}\right]\right\}$$

whence, in particular

$$(252\text{-}2)\quad C_0^{\theta}(t) = R_\omega(t)\frac{1}{u}\left\{\frac{1 - e^{-\left(\frac{1}{\Theta}-u\right)\theta}}{1 - u\Theta} - \left(1 - e^{-\frac{\theta}{\Theta}}\right)\right\}$$

and

$$(252\text{-}3)\quad C_\theta^{\infty}(t) = R_\omega(t)\frac{1}{u}\left\{\frac{e^{-\left(\frac{1}{\Theta}-u\right)\theta}}{1 - u\Theta} - e^{-\frac{\theta}{\Theta}}\right\}$$

or alternatively

$$(252\text{-}4)\quad C_\theta^{\infty}(t) = R_\omega(t)\left\{\frac{e^{u\theta}}{1 - u\Theta} - 1\right\}\frac{e^{-\frac{\theta}{\Theta}}}{u}$$

Clearly, in the light of the observations presented above in § 115, and from relation (123-9), the expressions for $C^{\theta_2}_{\theta_1,\omega}$, $C^{\theta}_{0,\omega}$ and $C^{\infty}_{\theta,\omega}$ can be obtained by writing $i=0$ in the preceding relations i.e. by putting $u=-\rho$.

For $\theta=0$, we have (relation 251-5)

$$C(t) = \frac{\Theta}{1-u\Theta} R_\omega(t) = \Theta_0 R_\omega(t) \tag{252-5}$$

which is relation (251-9) given above.

Case of a Stationary Process

253. The formulae corresponding to a stationary process can be deduced immediately from those just given by putting $\rho=0$.

In this case, of course, we have

$$\hat{\varphi}=\hat{\varphi},\quad \Theta=\Theta,\quad R=R_c,\quad \gamma=\gamma_c,\quad \bar{R}=\bar{R}_c \tag{253-1}$$ [1]

as before. Here the Capital Output Ratio γ is equal to the average amortisation period Θ of primary capital C_ω [2].

$$\gamma=\Theta$$
$$=\frac{C_\omega}{R_\omega}$$

[1] Relations (251-5), (251-6), (251-3), (251-4), (251-7), (251-8), (251-11), (251-12), (251-15) and (251-16).

[2] Relations (251-6), (251-10) and (253-1).

Part III

CONFRONTATION OF THE MODEL AND OSBERVED DATA

A. — THE GENERAL MODEL AND OBSERVED DATA

310. In my Tokyo monograph (1960) and my New York lecture (1961), I examined the justifications of the hypotheses made in, and the conclusions flowing from, the model whose main properties have been described in the preceding sections, both for the general case and for the case of exponential decline of the elasticity $\beta(\theta)$.

In the space at my disposal, it is not possible to cover these different points in detail, and I will do no more than recall the main features. The interested reader can find the details in my earlier publications. The discussion below also treats of several particularly interesting technical points which I have not previously had occasion to present.

For clarity's sake, the general model will be studied first.

1. Estimation of the parameters of the general model

1) *The Coefficient of Homogeneity* k

311. I suggested in my Tokyo monograph [1], and in my Econometrica article that there are three reasons for believing that

$$k \sim 1 \; . \tag{311-1}$$

[1] Allais (1960 A) § 39.

The evidence for this is derived from study of the ROSTAS analysis of 31 British and American industries in the 1930's [1], of the results derived by DOUGLAS and subsequent researchers and finally, the stability of economic equilibrium when suitable monetary conditions obtain.

2) *The function* $\psi(u)$

312. Unfortunately, in the present state of knowledge, it appears to be impossible to determine the function $\psi(u)$ of the general model with precision. Indeed, the only elements for which accurate numerical data are available on the basis of existing statistics are the ratios

$$\gamma = \frac{C}{R} \quad , \quad \frac{R}{R_\infty} \quad , \quad \frac{R_c}{R}$$

Now, it has already been shown [2] that for the model

$$\left\{\begin{array}{ll} \gamma = \dfrac{1 - \psi(i-\rho)}{i - \rho\,\psi(i-\rho)} & (312\text{-}1) \\ \\ \dfrac{R}{R_\infty} = \dfrac{i - \rho\,\psi(i-\rho)}{(i-\rho)\,\psi(i-\rho)} & (312\text{-}2) \\ \\ \dfrac{R_c}{R} = \dfrac{i-\rho}{i - \rho\,\psi(i-\rho)} & (312\text{-}3) \end{array}\right.$$

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(1) ROSTAS (1948).
(2) Relations (226-3), (225-3) and (225-2).

The expressions for these three ratios contain only the function $\psi(i - \rho)$, so that for any country at a given moment, there is in fact available *one and one only* piece of information about $\psi(u)$.

Eliminating $\psi(i - \rho)$ from these three expressions, we find

$$\frac{R}{R_\omega} = \frac{1}{1-\gamma i} \tag{312-4}$$

$$\frac{R_c}{R} = 1 - \gamma \rho \tag{312-5}$$

These two relations are in fact relations (123-6) and (123-7) which were developed in the examination of quasi-stationary process with a variable rate of growth of primary income, and *they are absolutely independent of hypothesis* (b) ([1]).

The first has general validity since it results from (110-2) *and* (110-9) (§ 226), *the second follows exclusively from the fact that we have simultaneously*

$$\begin{cases} \dfrac{1}{C} \dfrac{dC}{dt} = \rho & (312\text{-}6) \\[2ex] \dfrac{1}{R_\omega} \dfrac{dR_\omega}{dt} = \rho & (312\text{-}7) \end{cases}$$

([1]) § 211.

The same is true for all the relations which can be derived therefrom (1).

Estimation of Θ_o

313. If the *assumption* is made that the function $\psi(u)$ converges rapidly for small values of u, the relation

(313-1) $$\Theta_o = \frac{\gamma}{1 - i\gamma}$$ (2)

which neglects first order terms which include $i - \rho$ can be used to derive an estimate of the order of magnitude of Θ_o.

Estimation of δ

314. It would be very useful to have an estimate of δ, since, under the hypothesis of an exponential decrease of the function $\beta(\theta)$, we have $\delta = 0$ (3).

However, even if rapid convergence of the function $\psi(u)$ for small values of u is *assumed*, and admitting as a consequence that we can write (4)

(314-1) $$\psi(i-\rho) \sim 1 - \Theta_o(i-\rho) + (1+\delta)\Theta_o^2(i-\rho)^2$$

(1) This is notably the case for the interesting relation

$$\gamma_c - \gamma = \frac{\rho\gamma^2}{1 - \rho\gamma}$$

given above (125-2).

(2) Relation (226-3) and (240-11).

(3) Relation (251-1).

(4) Relation (240-1).

the knowledge of $\psi(i-\rho)$ on its own can be used to derive only a single relation between Θ_o and δ.

Thus, for example, the statistical knowledge of γ gives the relation [1]

$$\gamma = \frac{C}{R} \sim \Theta_o\left[1-\Theta_o\, i-\delta\, \Theta_o(i-\rho)\right] \tag{314-2}$$

In other words, there is only *one empirical relationship* available to determine the two parameters Θ_o and δ.

The condition [2]

$$\frac{\overline{R}_C}{\overline{R}_{CM}} \sim \left[1-\frac{\Theta_o^2}{2}\,(i-\rho)^2-\delta\,\Theta_o^2(i-\rho)^2\right]^k \tag{314-3}$$

from which a second expression could be derived, *is practically unusable,* for the variation of $\overline{R}_C/\overline{R}_{CM}$ with i and ρ is second-order and completely negligible by comparison with the influence of the other factors determining the level of real consumed income.

Alternatively, we could think of using one of the two relations [3] [4]

$$\Theta \sim \Theta_o\left[1-\Theta_o(i-\rho)-2\delta\,\Theta_o(i-\rho)\right] \tag{314-4}$$

$$\hat{\Theta} \sim \Theta_o\left[1-\Theta_o\, i-2\,\delta\,\Theta_o\, i\right] \tag{314-5}$$

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(1) Relation (240-11).
(2) Relation (240-13).
(3) Relation (240-5) and (240-6).
(4) We have $\Theta-\hat{\Theta} \sim (1+2\,\delta)\,\Theta_o^2\,\rho$.

but recourse to available statistical material to determine the functions $\varphi(\theta)$ and $\hat{\varphi}(\theta)$ directly, and using them to derive Θ and $\hat{\Theta}$ would involve such large estimating errors that the calculation of Θ_o and δ from the two preceding relations would be completely devoid of significance.

It is easy to see this. For, from relations (240-5) and (240-10) we have

$$(314\text{-}6) \qquad \gamma_C - \Theta \sim \delta\, \Theta^2_o\, (i - \rho)$$

so that

$$(314\text{-}7) \qquad \delta \sim \frac{\gamma_C - \Theta}{\Theta_o^2\,(i - \rho)}$$

or, putting $\Theta_o \sim \gamma_C$ as a first approximation

$$(314\text{-}8) \qquad \delta \sim \frac{\gamma_C - \Theta}{\gamma_C^2\,(i - \rho)}$$

It is obvious that, even where the relative error in Θ is relatively low, the relative error in the difference $\gamma_C - \Theta$ will be high. The same is true of the difference $i - \rho$ (1). So much

(1) An accurate estimate of i poses obvious difficulties; the same is true for ρ. We have

$$\rho = \frac{1}{R_\omega}\frac{d\,R_\omega}{dt}$$

An estimate of ρ in which this parameter is taken as being equal to the rate of population growth can clearly only be a first approximation. It follows that for small values of i and quite high values of ρ, the relative error in the estimate of $i - \rho$ can be fairly large.

so that in the existing state of knowledge, even if the *gratuitous* assumption of rapid series convergence is made, the possibility of making a serious estimate of the constant δ appears to be out of the question.

At the same time, it should be clearly understood that these difficulties do not arise from the model itself but are inherent in the nature of things.

2. JUSTIFICATION OF THE GENERAL MODEL WITH RESPECT TO ITS HYPOTHESES

General Model - Hypotheses (a), (b) *and* (c)

The general model is based on three hypotheses:

(*a*) Paretian optimum and invariance of the difference $i(t) - \rho(t)$,

(*b*) unchanging production elasticities and

(*c*) a steady rate of growth ρ of primary income R_ω (§ 210-211 and 230).

Hypothesis (a)

315. The assumption of a Paretian optimum can be accepted as a first approximation, *at least* insofar as equivalence over time of primary inputs and the comparison in time of their relative productivity is concerned.

The hypothesis that the difference $i(t) - \rho(t)$ is invariant is an asymptotic property of capitalistic processes *which can be demonstrated on the basis of very general assumptions* (1).

(1) ALLAIS (1962 A) p. 702, ALLAIS (1963) § 117 and 127 and ALLAIS (1964) § 53.

Hypothesis (b)

316. The assumption of logarithmic linearity is justified by the results obtained in the research undertaken elsewhere on production functions (1).

It is further justified by the fact that a general property of nature is that the empirically measured regularity of numerous phenomena can be expressed a wide range by logarithmically linear formulae. This explains, for example, the relative stability of demand elasticities, the approximative constancy of the rates of growth of population and production of different kinds, and the general applicability of the lognormal statistical distribution.

It is worth stressing at this point that *this hypothesis can only be considered as valid in the context of a finite range of variation of the parameters* (2) (3).

The invariance of $\beta(\theta)$ over time is an assumption which can have no a priori justification. *It can only be justified by its consequences.*

Hypothesis (c): *Stability of the Rate of Growth* ρ *of Primary Income* R_ω

317. The hypothesis of a constant value of the rate of growth ρ can be justified by the fact that if economic development is

(1) Allais (1960 A), § 37.

(2) In discrete notation if the production function (note 2, p. 45, § 211)

$$\overline{R} = a\, R_0^{b_0}\ ,\ R_1^{b_1}\ ,\ \ldots,\ R_0^{b_0}\ ,\ \ldots$$

were considered valid for any value of primary input, the manifestly unacceptable conclusion would follow that $\overline{R}$ would necessarily be zero if any one R_0 were zero.

(3) Naturally, hypothesis (*b*) implies the more general hypothesis according to which there exists a valid index $\overline{R}_c$ of the real national consumed income (§ 115, 119 and 211).

considered in terms of trends, the rates of growth appear as approximately constant over large segments of time.

At all events, it must be stressed that the great majority of the results are obtained independently of this hypothesis [1]. The hypothesis of constancy of ρ enters into the culculation of $\hat{\varphi}$, $\hat{R}_{\omega}$, $\hat{\Theta}$ and $\overline{R}_{CM}$ [2] only.

3. Justification of the general model with respect to its consequences

There are at least two points on which the general model appears to be justified by the conclusions to which it leads.

a) *Weak Relationship between Real Income per Capita and the Capital-Output Ratio at any Given Moment*

318. We have seen that

$$\frac{\overline{R}_C(t)}{\overline{R}_{CM}(t)} = \left[\frac{1-\gamma\rho}{1-\gamma i} \; e^{-\Theta_0 (i-\rho)} \right]^k \qquad (318\text{-}1)$$

where $\overline{R}_{CM}(t)$ is independent of the capitalistic structure [3].

For $k \sim 1$, $\Theta_0 \sim 4$, and i and ρ of the order of a few per-cent, this equation shows the relationship between real income

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(1) § 220 to 229.
(2) § 230 to 233.
(3) Relation (228-14). $\overline{R}_{CM}(t)$ is given by (228-13) and if ρ is constant by (233-2) and (233-3).

per head and the capital-output ratio $\gamma = C/R$ to be a very weak one *in different countries over a given time span,* and this is in conformity with observed facts.

In practise, no statistical interdependence can be found between real income per capita and the capital-output ratio γ. While this coefficient has real influence, it is insignificant by comparison with the influence of other factors.

It can thus be seen that the theory presented leads to the expectation of, and provides an explanation for, a weak relationship between real income per capita and the capital output coefficient.

Such a finding demonstrates only the consistency of the model with observed facts. *It does not prove that the assumption of logarithmic linearity of the production function is necessarily correct.* It is evident that there exists an infinite number of production functions which imply a low interdependence of real income and the capital-output ratio.

b) *Weak Relationship over Time between Real Income per Capita and the Capital-Output Ratio*

319. *As the coefficient* Θ_0 *is assumed to be constant over time, the preceding formula leads to the expectation of a low relationship over time between real income per capita and the capital-output ratio,* and again this corresponds to observation.

But here again, no conclusion can be drawn other than that the model is consistent with reality.

B. — THE EXPONENTIAL MODEL AND OBSERVED DATA

1. Estimation of the parameters of the exponential model

320. The exponential model contains only two parameters, k and Θ_o.

The estimate of the value of k is derived in the same way as in the general model.

Estimation of Θ_0

321. We have found the relation

$$\Theta_0 = \frac{\gamma}{1 - i\gamma} \tag{321-1}$$

with $\gamma = C/R$ (1).

Thus Θ_0 depends only in γ and i and this relation holds whatever the value of ρ.

Methods of estimating γ and i are given in my Tokyo monograph and in my New York lecture (2) (3).

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(1) Relation (251-13).
(2) Allais, 1960 A, § 42.
(3) Allais, 1962 A, pp. 714-715.

Here are the main results (¹):

TABLE 4

Estimates of Θ_0			
Averages	U.S. - G.B. - France - 1913		4.41
	United States	1880-1900 1906-1913 1950-1956	4.34 4.11 3.64
		Average 1880-1956	4.12

The estimates of Θ_0 are of the same order of magnitude for the United States from 1880 to 1956, and for the United States, France and Great Britain in 1913. Θ_0 is roughly 4.

2. Justification of the exponential model with respect to its hypothesis

Hypothesis (d): *The Relative Productivity of Primary Inputs*

322. Hypothesis (*d*) (²) cannot be confirmed directly from experience, and can only be justified by the concordance of the conclusions flowing from it with observed data.

However, a priori, it seems intuitively reasonable to assume at least as a first approximation that production elasticities decrease exponentially with time.

Further, this assumption is closely related to hypothesis (*b*) of invariance of the function $\beta(\theta)$ over time. In fact, postulating the invariance of the capitalistic production function of the productive system over time implies not merely the constancy of the function $\beta(\theta)$, but, also, if we think it over, the fact

(¹) See § 324 below.
(²) § 250.

that the relative productivity of two primary inputs depends only on their distance from each other in time, and this is exactly what hypothesis (*d*) says.

3. Justification of the model with respect to its consequences

The main points on which conclusions derived from the exponential model are justified are the following:

a) *Small Inter-country Differences in the Value of the Capital-Output Ratio at the Same Point of Time*

323. The relation

$$\gamma = \frac{\Theta_0}{1+\Theta_0 i} \qquad (323\text{-}1)$$ [1]

shows that if Θ_o as a technical constant applies to the different countries considered, the capital-output ratio γ should change but little, for according to the formula such changes cannot be very great at observed rates of interest and with a value of Θ_o of about 4.

Thus, for 1913, I have calculated Θ_o as [2]:

United States	3.44
France	3.83
Great Britain	3.67
Average	3.64

(1) Relation 251-6.
(2) Allais (1960 A), p. 715; (1962 A), pp. 52-53.

b) *Low Variability over Time of the Capital-output Ratio γ within the same Country*

324. *According to the assumption made, the function $\beta(\theta)$ is in variant over time, whence it follows that Θ_0 is a constant, independent of time* (1). This leads to the conclusion that for any given country, changes in γ are the result solely of changes in i, which means that γ should vary but little over time for any one country (2).

The record shows that this is indeed the case for the U.S.A. where, considering years in which comparable levels of full employment were attained, the coefficient γ takes the values shown below (3).

TABLE 5

Year	γ	Period	Period Average	Long Period
1880 1890 1900	3.13 3.87 3.69	1880-1900	3.56	
1906 1910 1913	3.37 3.38 3.44	1906-1913	3.40	1880-1913: average: 3.48
1923 1929 1937	3.34 3.46 3.88	1923-1937	3.56	1880-1956: average: 3.46
1950 1955 1956	3.40 3.30 3.35	1950-1956	3.35	

(1) § 211, 250 and relation (250-5).
(2) Relation (323-1).
(3) ALLAIS, 1962 A, p. 714.

The exponential model thus leads to the expectation that the value of the capital-output ratio will differ little across countries at different times.

If we use COLIN CLARK's figures, consisting of 58 estimates for 21 countries for different dates between 1805 and 1953, we find that they are lognormally distributed with a median value of 3.54 (1).

The overall results are therefore as follows:

U.S.A., 1880-1956 (median of 12 observations) . .	3.46
U.S.A., France, U.K., 1913 (average)	3.72
World - 1805-1953 (median of 58 observations) . .	3.54

The concordance of these different estimates is absolutely remarkable. *It suggests a temporal and spatial regularity which in any case must be explained.*

The explanation given by the model is a simple one.

c) *Constancy of the Ratio* C/R_ω

325. The exponential model contains a very remarkable relationship:

$$C(t) = \Theta_o \, R_\omega(t) \tag{325-1}$$ (2)

According to this expression, statistical analysis should show an approximative stability over space and in time of the ratio of capital to primary income.

(1) ALLAIS (1960 A), pp. 54-57; (1962 A), p. 723.
(2) Relation (251-9).

Since *for any economy* (1)

$$R = R_{\omega} + i\,C \tag{325-2}$$

it follows that we have

$$\frac{C}{R_{\omega}} = \frac{C}{R - iC} = \frac{\gamma}{1 - i\gamma} \tag{325-3}$$

for any economy.

In other words, testing that the ratio C/R_{ω} remains equal to a constant Θ_o is eqnivalent to testing that the ratio $\gamma/(1 - i\,\gamma)$ remains equal to this constant.

This is precisely what was done above in estimating Θ_o (§ 321).

d) *Exponential Form of the Amortization Function* $\varphi(\theta)$

326. The exponential amortization relationship

$$\varphi(\theta) = \frac{1}{\Theta}\, e^{-\frac{\theta}{\Theta}} \tag{326-1}$$ (2)

(1) Relation (110-2).
(2) Relation (251-3).

derived from the exponential model appears to concord very well with the results obtained from analysis of available data on durable goods in all cases where a sufficiently large competitive market in these goods exists (1) (2).

An attempt to derive the fonction $\varphi(\theta)$ directly, at least approximatively, can be made with the help of statistics of the distribution of the working population. This analysis involves major difficulties, but at first sight seems to produce results in conformity with an exponential form of $\varphi(\theta)$ (3).

The length of the production period $\hat{\Theta} = \gamma$ is of the order of 3.5 years for the U.S.A. For the average amortisation period $\Theta = \gamma_C$, from (251-6)

(326-2)
$$\Theta = \frac{\hat{\Theta}}{1 - \hat{\Theta}\rho}$$

Whence, for $\hat{\Theta} = 3.5$ and $\rho = 1.7\%$ (4), Θ is derived for the U.S.A. as 3.72, i.e. a value which differs little from that of $\hat{\Theta}$.

(1) ALLAIS (1960 A), p. 22-23.

(2) It may be objected that in a rigorous formulation, the composition of different exponential amortization functions cannot result in an exponential global amortization function over the whole range of variation (0, ∞) of θ. Nevertheless, over the useful variation interval (0, 100) of θ, use of this type of function is certainly possible, as it involves a *rather low* relative error.

(3) ALLAIS (1960 A), § 45.

(4) Equal, as a first approximation, to the rate of population growth in the U.S.A.

C. — GENERAL COMMENTS

Constancy of Θ_0

330. The observed stability over both time and space of the coefficient Θ_o may at first sight seem extraordinary. However, on further reflection, and in the framework of the model, this stability becomes much less astonishing. It expresses the fact that whatever the state of knowledge, the decrease in the productivity of the factors of production as they become more distant in time remains constant. *In a certain sense, this is an index of the conceptual difficulty involved in thinking about roundabout processes* (1).

Estimation of Θ and the Distribution of the Active Population

331. There is a manifest relation between the value of the average amortization period Θ and the composition of the working population, and it would seem reasonable to envisage estimating the value of Θ directly in order to compare it with its theoretical value, which in the case of the exponential model is

(331-1) $$\Theta = \Upsilon_c$$ (2)

(1) Relation (211-0) and (251-2).
(2) Relation (251-11)

Any hope of doing this is illusory, since it has been shown *that whatever the function* $\beta(\theta)$, for small values of i and φ we have

$$(331\text{-}2) \qquad \Theta \sim \gamma_c \qquad (^1)$$

In practise, the rates i and ρ are quite small and it is therefore practically *certain* that the order of magnitude of the value of Θ should be about the same as γ_C. *But if this concordance does occur in fact, it only shows that the reasoning based on the composition of the working population is correct; in no way does it demonstrate the validity of the model proposed.* This demonstration would only be possible if the determination of the function $\varphi(\theta)$, and therefore of Θ, from the composition of the working population, could be done accurately enough for the difference $\gamma_C - \Theta$ itself to be determined sufficiently precisely. As has already been noted, this is an undertaking which appears to be impossible, at least in the present state of knowledge (2).

The Influence of the Function $\beta(\theta)$

332. It is interesting to examine what the results which correspond to the exponential model become when a non-exponential expression is used for $\beta(\theta)$.

Appendix I contains a study of the case in which the quite acceptable hypothesis is postulated that $\beta(\theta)e^{\mu\theta}$ can be repre-

(1) Relation (227-6). For $i=\rho=0$, we have

$$\Theta = \gamma$$

(2) § 312 to 314.

sented by a convergent Taylor series for any value of θ. It is worth remarking that in most of the cases which can reasonably be envisaged, the coefficient δ remains relatively small, where upon the properties of the model are quite similar to those of the exponential model.

Amortization of Primary Income

333. The relationship between the amortisation function $\varphi(\theta)$ and the usually accepted notion of amortisation raises a number of problems to which allusion has already been made (¹). It is of interest to examine these problems in the light of the results which have been obtained.

1) In the first place, the usual definition considers investment as occurring when an already completed investment good begins its productive career. By contrast, the act of investment is considered in this paper as taking place when the primary inputs are furnished. A distinction must therefore be made between the amortisation of primary income and the amortisation of investment. It further follows that under the usual definition, amortisation contains an element of interest as well as the depreciation of primary inputs, whereas in the present paper, amortisation relates only to the primary input content of the investment.

2) A part of the primary income invested at a given instant t emerges in the global producion P^* of consumption and investment goods at instant $(t+\theta)$. The primary inputs which correspond to the latter are thus reintroduced into the

(¹) § 116.

production process. Some portion of them will reappear at instant $(t + \theta + \theta')$ in the global production of that instant. The primary inputs which correspond to the durable goods element of this production again enter the production circuit, and so on.

There is therefore a close relationship between the amortisation schedule of primary income $\varphi(\theta)$ *and the amortisation schedule for durable goods* from the time they emerge from the production process and are themselves put into operation to produce further new production of consumption goods. Clearly, of course, the amortisation schedule for primary income is not identical to the durable goods schedule.

The supply of reproducible capital at a given moment is composed of three elements:

a) Inventories (stocks and work in progress);

b) Equipment (machines, lorries, motor cars, washing machines, etc.);

c) Structural assets (factories, houses, roads, etc.).

No item of primary income arising at any given moment can emerge in subsequent consumption unless it has been incorporated during the interval either in inventories (stocks and work in progress) or in equipment or structural investment. There is thus a close relationship between the amortisation function of primary income $\varphi(\theta)$ and the structure of reproducible capital.

Finally, since inputs of labour constitute a part of primary income there is also a close link between the amortisation function of primary income $\varphi(\theta)$ and the composition of the labor force which corresponds at any given moment to the allocation of the amounts of labour available as between the different phases of the production process. In particular, as has been noted earlier, the invariance of the function $\varphi(\theta)$ over time follows from the invariante of the composition of the labor force [1].

Any assumption concerning the amortisation function $\varphi(\theta)$ of primary income can therefore be tested against observed reality from three points of view:

a) The composition of global production P* in terms of consumption goods and investment goods, and the amortisation schedule for investment goods and durables;

b) The structure of reproducible capital;

c) The composition of the labor force.

3) The shape of the amortisation schedule $\varphi_i(\theta)$ for any particular economic sector i generally bears little relation to an exponential curve, at least for sufficiently low values of θ. $\varphi_i(\theta)\,d\theta$ represents the share of primary income which will emerge in consumed income during the interval θ to $\theta+d\theta$. But, taking steel manufacture as an example, the industry's output can only enter final consumption after a period of at least some months; it must first pass through a number of intermediate stages. It follows that primary income absorbed by the steel industry can only emerge into final consumption after a certain lapse of time. The function $\varphi(\theta)$ thus starts with a value of zero, grows, reaches a maximum and then declines. The same is true of the majority of economic sectors [2].

Naturally,

(333-1) $$R_\omega \varphi(\theta) = \sum_i R_{\omega,i}\ \varphi_i(\theta)$$

where $R_{\omega,i}$ is the primary income devoted to sector i.

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(1) § 120.

(2) The shape of the majority of $\varphi(\theta)$ functions is therefore similar to that of the function $\varphi(\theta)$ which corresponds to the para-exponential model examined in annex (Section II A, § 520 to 524).

In the present state of knowledge, it is difficult, if not impossible, to assign a particular form to the $\varphi_i(\theta)$ functions. At the same time, the analysis in the present study provides numerous reasons for believing that the shape of function $\varphi(\theta)$ should not differ too greatly from an exponential form.

4) Global production P^* at any given moment consists of the total of consumed income R_C and investment I:

(333-2) $$P^* = R_C + I\ .$$

But on the cost side,

(333-3) $$P^* = R_\omega + \mathfrak{A} + iC$$

where primary income R_ω includes the wage bill and rent from ownership of property, iC represents the interest charged on the global value of the existing reproducible capital, and $\mathfrak{A}$ represents the effective amortisation of reproducible capital defined by the relation

(333-4) $$\mathfrak{A} = I - \frac{dC}{dt}$$

Thus, we have

(333-5) $$P^* = R_C + I = R_\omega + \mathfrak{A} + iC$$ [1]

[1] See ALLAIS, *Les Fondements Comptables de la Macroéconomique* (The Accounting Bases of Macroeconomics).

5) Global production P^* corresponds exactly to gross national product P' as conventionally defined if the two following adjustments are made:

a) In the first place, the quantity

$$(i - \rho_C)\ C_C \qquad (^1)$$

should be added to P', where C_C is the overall value of existing stocks of durable consumption goods, i the rate of interest, and ρ_C the rate of growth C_C. Valuation is in terms of the unit of primary income, i.e. the basic hourly wage-rate (²).

b) The Gross National Product, P', as usually defined covers private consumption, private investment and government purchases of goods and services. But a part only of government expenditure should be considered as corresponding to final collective services rendered directly to the public; the outlays which correspond to services rendered to the business sector should be deducted. As a first approximation, this quantity can be taken as taxes paid by business (³).

Thus the relationship between P and P' is:

$$(333\text{-}6) \qquad P^* = P' + (i - \rho)\ C_C - \mathcal{T}_E$$

when $\mathcal{T}_E$ represents the taxes paid by business.

(¹) § 116.

(²) § 110.

(³) On this point, see ALLAIS, *Les Fondements comptables de la macroéconomique* (The Accounting Bases of Macroeconomics), op. cit., pp. 48-49 and 54-55, and KUZNETS, *Government Product and National Income*, Income and Wealth, Serie I, Bowes, Cambridge, pp. 178-194.

6) The estimates of capital-output ratios γ and γ_C given in § 320 and § 326 have been calculated on the basis of available figures for national income. In point of fact, the compilation of these figures is subject to many conceptual difficulties, and the data must be adjusted if they are to be used for even a midly rigorous application of formulations in the present study. It may be helpful at this point, to illustrate the problem by examining data for the U.S.A. economy in 1956.

In that year, it is possible to take as a first approximation:

$$i = 2.55\% \quad (1)$$

$$\rho = 1.7\% \quad (2)$$

$$\rho_C = \rho = 1.7\% \quad (3)$$

$$i - \rho = i - \rho_C = 0.85\%$$

(1) i here is the pure rate of interest *in wage units*, since the unit of valuation in the present study is the unit of primary income i.e. the basic hourly wage rate (§ 110). In my New York lecture, published in « Econometrica » in 1962, I calculated

$$i = i_n - \frac{\sigma}{2} = 4.57 - \frac{3.95}{2} = 2.55$$

where i_n is the pure rate of interest in nominal terms and σ the rate of growth of hourly wage rate (p. 714).

(2) § 326.

(3) Since the composition of reproducible capital is fairly stable over time, as a first approximation, it is possible to take

$$\frac{1}{C_C}\frac{dC_C}{dt} \approx \frac{1}{C}\frac{dC}{dt}$$

and, in billions:

$$P' = 419.2 \qquad (^1)$$

$$C = 1199.7 \qquad (^2)$$

$$C_C = 163.4 \qquad (^3)$$

$$\mathcal{T}_C = 35.7 \qquad (^4)$$

Then, from (333-6),

$$P^* = 419.2 + 0.0085 \times 163.4 - 35.7 = 384.9 \ .$$

Private investment (excluding consumer durables) amounted to \$64.4 billion (5). To this figure should be added purchases of consumer durables amounting to \$38.5 billion (6), making a total for private investment outlays of \$105.9 billion. The value of public investment is not specified, but it may be assumed that both kinds of investment are in proportion to their share of existing non-monetary reproducible capital. This enables the value of investment to be estimated as

$$I = \frac{1199.7}{1199.7 - 156.3} \times 105.9 = 121.8 \qquad (^5)$$

(1) « Economic Almanac », 1962, p. 121.

(2) Excluding monetary metals (« Economic Almanac », 1962, p. 140) valued at \$26.5 billion:

$$1226.2 - 26.5 = 1197.7 \ .$$

The figure of 1226.2 corresponds to the global value of reproducible capital, of which 26.5 represents the value of monetary stocks of gold and silver.

There is also a question as to whether military assets (\$84.3 billion) should not be included also. These assets contribute to the supply of the final collective service « defense ». On balance, it seemed preferable to omit them, but it is open to discussion whether they should not have been counted in.

(3) Ibid., p. 140.

(4) Indirect business Tax, Ibid., p. 121.

(5) « Economic Almanac », 1962, p. 122.

(6) Ibid., p. 140. In billions of dollars.

Public buildings and equipment $= 5.6 + 145.7 + 5 = 156.3$.

The figure of 156.3 corresponds to the overall value of the different classes of governmental reproducible capital, excluding military assets valued at 84.3.

Correspondingly, from (333-5)

$$R_C = P^* - I = 384.9 - 121.8 = 263.1$$

whence

$$\gamma_C = \frac{C}{R_C} = \frac{1199.7}{263.1} = 4.56$$

Again, in billions of dollars:

Valne of inventories = 122.3 [1]

Value of equipment = 340.8

Value of structures = 736.6

So that the value χ of the share of reproducible capital accounted for by stocks and goods in course of production can be calculated as

$$\chi = \frac{122.3}{1199.7} = 0.102$$

Since

(333-7) $$R_\omega = [1 - \gamma_C(1 - \rho)]R_C$$ [2]

(333-8) $$dC/dt = \rho C$$ [3]

(333-9) $$\mathcal{A} = I - \frac{dC}{dt} = I - \rho C$$ [4]

(333-10) $$R = R_C + \rho C$$ [5]

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(1) « Economic Almanac », 1962, p. 140.

(2) Relations (110-1), (110-2) and (123-2). It may be recalled that the relation (123-2) is based on the two assumptions that $(i - \rho)$ is constant and that the function $\varphi(0)$ is invariant over time, i.e. that the composition of the labor force does not change (120). Both assumptions are fairly weak.

(3) Relations (251-9) and (110-6).

(4) Relations (333-4) and (333-8).

(5) Relations (110-1) and (333-8).

it can therefore be deduced that

$$R_\omega = (1 - 4.56 \times 0.0085)\ 263.1 = 252.9$$

$$dC/dt = \rho C = 0.017 \times 1199.7 = 20.4$$

$$\mathfrak{A} = 121.8 - 20.4 = 101.4 \qquad (^1)$$

$$R = 263.1 + 20.4 = 283.5$$

$$iC = 0.0255 \times 1199.7 = 30.6$$

For the U.S.A. in 1956, relation (333-5) above

$$P^* = R_C + I = R_\omega + \mathfrak{A} + iC$$

can finally be written

$$384.9 = 263.1 + 121.8 = 252.9 + 101.4 + 30.6\ .$$

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(1) This figure differs greatly from the 34.4 given in the « Economic Almanac », 1962, p. 12, which related however only to private investment. On the assumption that the relative values of amortisation of consumer durables and government investment are the same the 34.4 becomes

$$34.4 \cdot \frac{1199.7}{1199.7 - 319.7} = 46.9$$

The 319.7 is derived as the total of 163.4 (consumer durables) and 156.3 (public investment). The difference between the two figures 101.4 and 46.9 is a reflection of the inadequacy of the usual methods of estimating amortisation. Whatever calculating process is adopted, it should respect the identity

$$I = \mathfrak{A} + \frac{dC}{dt}$$

on which I have based the calculation of $\mathfrak{A}$.

It follows from this relation that primary income R_ω is allocated both to consumption and investment goods, and as a first approximation it may be assumed that the share of amortisation and interest in R_C and I is the same (1). On this basis, the position can be summarised as in the following table:

TABLE 7

U.S.A. 1956					
	R	$\mathfrak{a}$	*i*C	Total	%
R	172.9	69.3	20.9	263.1	68.4
I	80.0	32.1	9.7	121.8	31.6
Total	252.9	101.4	30.6	384.9	100
%	65.7	26.3	8.0	100	

The table shows that under the assumptions made, the share of primary inputs in investment at any given moment corresponds to

$$\lambda = \frac{I}{R_c + I} = 0.316$$

only some 32% of primary income, whereas they represent

$$1 - \chi = 0.898$$

in the region of 90% of reproducible capital. This quite remarkable circumstance calls for explanation.

(1) See relations (251-8), (251-9) and § 251 *in fine*. See also Tables 8 and 9 below.

The Case of the Exponential Model

334. In the case of the exponential model, it is possible to precise these different indications:

1) In the first place,

(334-1) $$\Theta = \gamma_C = C/R_C$$ (1)

(334-2) $$\hat{\Theta} = \gamma = C/R$$ (2)

(334-3) $$\Theta_0 = \frac{\Theta}{1 - (i - \rho)\Theta}$$ (3)

(334-4) $$\hat{R}_\omega = \frac{1 + \Theta_0(i - \rho)}{1 + \Theta_0 i}$$ (4)

whence

$$\Theta = \gamma_C = 4.56$$

$$\hat{\Theta} = \gamma = 1199.7/283.5 = 4.23$$

$$\Theta_0 = \frac{4.56}{1 - 0.0085 \times 4.56} = 4.74$$

$$\hat{R}_\omega = \frac{1 + 0.0085 \times 4.74}{1 + 0.0255 \times 4.74} \times 252.9 = 234.7$$

(1) Relation (251-11).
(2) Relation (251-12). Also, from (251-6)

$$\hat{\Theta} = \Theta_0/(1 + \Theta_0 i) .$$

(3) Relation (251-5).
(4) Relation 251-8).

It can thus be seen that if the adjustment items $(i-\rho)C_C$ (upwards) and $\mathcal{I}_E$ (downwards) on R_C are taken into account, the values of Θ, $\hat{\Theta}$ and Θ_0 are some 15% higher than when the calculation is based on uncorrected figures of national income (1) (2).

2) In the case of the exponential model, it is of interest to distinguish the contribution of the different primary inputs to primary income, consumed income, capital and primary capital, according to the period in which they appear in consumed income. For illustration purpose, three intervals of time $(0, \theta_1)$, (θ_1, θ_2) and (θ_2, ∞) will be considered.

The shares of primary income which correspond to the intervals $(0, \theta_1)$, (θ_1, θ_2) and (θ_2, ∞) are respectively (3).

$$R_{0,\omega}^{\theta_1} = \int_0^{\theta_1} \varphi(\theta)\, d\theta = 1 - e^{-\frac{\theta_1}{\Theta}} \tag{334-5}$$

$$R_{\theta_1,\omega}^{\theta_2} = \int_{\theta_1}^{\theta_2} \varphi(\theta)\, d\theta = e^{-\frac{\theta_1}{\Theta}} - e^{-\frac{\theta_2}{\Theta}} \tag{334-6}$$

$$R_{\theta_2,\omega}^{\infty} = \int_{\theta_2}^{\infty} \varphi(\theta)\, d\theta = e^{-\frac{\theta_2}{\Theta}} \tag{334-7}$$

(1) § 320 to 326.

(2) If military assets ($84.3 billion) are included in the evaluation of reproducible capital (see note (2) p. 139 above), higher values still are reached:

$$\Theta = 5.14 \qquad \hat{\Theta} = 4.72 \qquad \Theta_0 = 5.37$$

but the inclusion of these assets when valuing capital is open to question.

(3) Relation (251-3). It will be recalled that:

$$\int_0^{\infty} \varphi(\theta)\, d\theta = 1 .$$

Clearly, consumed national income can be divided up into the corresponding shares, for which the expressions are, from

$$(334\text{-}8) \qquad R_{0,c}^{\theta_1} = R_\omega(t)\int_0^{\theta_1} e^{u\theta}\varphi(\theta)d\theta = \frac{1}{1-\Theta u}\left[1-e^{-\left(\frac{1}{\Theta}-u\right)\theta_1}\right]$$

(334-9)

$$R_{\theta_1,c}^{\theta_2} = R_\omega(t)\int_{\theta_1}^{\theta_2} e^{u\theta}\varphi(\theta)d\theta = \frac{1}{1-\Theta u}\left[e^{-\left(\frac{1}{\Theta}-u\right)\theta_1} - e^{-\left(\frac{1}{\Theta}-u\right)\theta_2}\right]$$

$$(334\text{-}10) \qquad R_{\theta_2,c}^{\infty} = R_\omega(t)\int_{\theta_2}^{\infty} e^{u\theta}\varphi(\theta)d\theta = \frac{1}{1-\Theta u}\left[e^{-\left(\frac{1}{\Theta}-u\right)\theta_2}\right]$$

with (¹)

$$(334\text{-}11) \qquad R_c = R_{0,c}^{\theta_1} + R_{\theta_1,c}^{\theta_2} + R_{\theta_2,c}^{\infty} = \frac{R_\omega(t)}{1-\Theta u}$$

and

$$(334\text{-}12) \qquad u = i - \rho .$$

(¹) Relation (251-7).

From relation (252-1), the corresponding shares for capital are

$$C_0^{\theta_1} = \frac{R_\omega}{u} \left\{ \frac{1 - e^{-\left(\frac{1}{\Theta} - u\right)\theta_1}}{1 - u\Theta} - \left(1 - e^{-\frac{\theta_1}{\Theta}}\right) \right\} \tag{334-13}$$

$$C_{\theta_1}^{\theta_2} = \frac{R_\omega}{u} \left\{ \frac{e^{-\left(\frac{1}{\Theta} - u\right)\theta_1} - e^{-\left(\frac{1}{\Theta} - u\right)\theta_2}}{1 - \Theta u} - \left[e^{-\frac{\theta_1}{\Theta}} - e^{-\frac{\theta_2}{\Theta}} \right] \right\} \tag{334-14}$$

$$C_{\theta_2}^{\infty} = R_\omega \left[\frac{e^{u\theta_2}}{1 - \Theta u} - 1 \right] \frac{e^{-\frac{\theta_2}{\Theta}}}{u} \tag{334-15}$$

u is given by (334-16), whence

$$C = C_0^{\theta_1} + C_{\theta_1}^{\theta_2} + C_{\theta_2}^{\infty} = \left[\frac{1}{1 - u\Theta} - 1 \right] \frac{R_\omega}{u} \tag{334-16}$$

$$= \frac{\Theta}{1 - u\Theta} R_\omega = \Theta_0 R_\omega \quad (^1)$$

Clearly, from the discussion in § 252, the corresponding shares of primary capital

$$C_{0,\omega}^{\theta_1} \qquad C_{\theta_1,\omega}^{\theta_2} \qquad C_{\theta_2,\omega}^{\infty}$$

(1) Relations (251-9) and (251-5).

can be obtained by putting $i=0$ in the expressions for the breakdown of capital i.e. by putting

$$u = -\rho \tag{334-17}$$

in the three relations (334-13) to (334-15).

Finally, the average amortisation periods for the three groups considered are

$$\Theta_0^1 = \frac{\int_0^{\theta_1} \theta\,\varphi(\theta)\,d\theta}{\int_0^{\theta_1} \varphi(\theta)\,d\theta} = \Theta - \frac{\theta_1 e^{-\frac{\theta_1}{\Theta}}}{1-e^{-\frac{\theta_1}{\Theta}}} \tag{334-18}$$

$$\Theta_1^2 = \frac{\int_{\theta_1}^{\theta_2} \theta\,\varphi(\theta)\,d\theta}{\int_{\theta_1}^{\theta_2} \varphi(\theta)\,d\theta} = \Theta + \frac{\theta_1 e^{-\frac{\theta_1}{\Theta}} - \theta_2 e^{-\frac{\theta_2}{\Theta}}}{e^{-\frac{\theta_1}{\Theta}} - e^{-\frac{\theta_2}{\Theta}}} \tag{334-19}$$

$$\Theta_2^\infty = \frac{\int_{\theta_2}^{\infty} \theta\,\varphi(\theta)\,d\theta}{\int_{\theta_2}^{\infty} \varphi(\theta)\,d\theta} = \Theta + \theta_2 \tag{334-20}$$

The last of these expressions is particularly simple. Finally

$$\Theta_{\tau_1}^{\tau_2} = C_{\tau_1}^{\tau_2} / R_{\tau_1}^{\tau_2} \qquad \text{for} \quad u=0 \tag{334-21}$$

3) As a further illustration, take

$$\theta_1 = 1 \text{ year} \qquad \theta_2 = 5 \text{ years}$$

From the estimates presented above

$$\Theta = 4.56 \qquad u = i - \rho = 0.85\% \qquad \rho = 1.7\%$$

and the following table may be constructed from relations (334-5) to (334-20).

TABLE 8

Characteristics for $\Theta = 4.56$ $u = 0.85\%$ $\rho = 1.7\%$ $\theta_1 = 1$ $\theta_2 = 5$ Exponential hypothesis	Breakdown of R_c, R_ω, C and C_ω			
	$0<\theta<1$	$1<\theta<5$	$0<\theta<5$	$5<\theta<\infty$
Share of Consumed Income $R^{\tau_2}_{\tau_1,c} / R_c$	0.190	0.461	0.651	0.349
Share of Primary Income $R^{\tau_2}_{\tau_1,\omega} / R_\omega$	0.197	0.469	0.666	0.334
Share of Capital $C^{\tau_2}_{\tau_1} / C$	0.020	0.272	0.292	0.708
Share of Primary Capital $C^{\tau_2}_{\tau_1,\omega} / C_\omega$	0.022	0.293	0.315	0.685
Capital Output Ratio $C^{\tau_2}_{\tau_1} / R^{\tau_2}_{\tau_1}$	0.480	2.691	2.045	9.251
Average Amortization Period $\Theta^{\tau_2}_{\tau_1}$	0.48	2.71	2.05	9.56

[11] *Allais* - pag. 148

It can be seen that primary inputs emerging in consumed income before a year has elapsed represent 20% of primary income, but only 2% of capital. On the other hand, inputs which appear in consumed income after 5 years represent 71% of capital, but 33% only of primary income.

It will also be observed that by reason of the low values of ρ and $(i-\rho)$ the difference between the shares corresponding to consumed income and primary income remains relatively small; this is also true of capital and primary capital.

For the same reason, the amortisation periods $\Theta_{\tau_1}^{\tau_2}$ differ relatively little from the ratios $C_{\tau_1}^{\tau_2}/R_{\tau_1}^{\tau_2}$.

Finally, it can be seen that for small values of θ_1, the contribution of primary income to capital is practically negligible. If (334-13) is developed as a Taylor series, the main component of $C_0^{\theta_1}/C$ is seen to be $\theta_1^2/2\,\Theta^2$.

4) These theoretical results help to explain the empirical figures obtained earlier [1] showing that the primary income incorporated in investment at any given moment represents 32% of the total primary income of that moment, whereas the contribution of the different inputs of primary income over time to reproducible capital is of the order of 90% [2].

5) To make an approximate assessment of the relationship between the empirical results and those in Table 8, we adopt the following hypothesis (H), bearing in mind that it is not rigorously verified, and indeed, can at best be only a rough assumption. The hypothesis is that primary inputs incorporated in investment can only emerge in consumed income after the expiry of a period θ', whereas the primary inputs which enter stocks and goods in course of production

(1) § 333-6.
(2) Each contribution representing 32% of the primary income of that moment.

all appear in consumed income before the end of θ'. The approximate nature of this hypothesis is evident from the fact that certain primary inputs which are incorporated in reproducible capital in the form of maintenance outlays or repairs appear in consumed income immediately, while at the same time the primary inputs which correspond to stock formation may only reach consumed income long after the inputs corresponding to certain types of investment.

If hypothesis (H) is taken as correct (which, rigorously, it is not), the contribution of primary income to investment, from the figures in table 8 and relation (334-15) should be

$$R^{\infty}_{\theta',\omega} = e^{-\frac{\theta'}{\Theta}} = \lambda = 0.316$$

i.e., for $\Theta = 4.56$

$$\theta' = 5.25 \qquad (^1)$$

From the preceding discussion, this certainly is an overestimate.

Similarly, if hypothesis (H) were true, the contribution of primary income to fixed investment (i.e. capital immobilised in the form of equipment and structures) should be

$$\frac{C^{\infty}_{\theta'}}{C} = \left[1 - \frac{1 - e^{\theta' u}}{\Theta u}\right] e^{-\frac{\theta'}{\Theta}} = 1 - \chi$$

For

$$\chi = 0.1 \qquad \Theta = 4.56 \qquad u = 0.0085$$

(1) See Chart II below.

the value of θ' is determined as

$$\theta' = 2.46 \qquad (^1)$$

The earlier discussion suggests that this estimate is an understatement. Thus, if hypothesis (H) were verified, we should have

$$2.46 < \theta' < 5.25 .$$

Since the exponential model is assumed to correspond to reality, the distance between the two values for θ' results from the incorrectness of hypothesis (H); but the fact that these two values are of the same order of magnitude does suggest that hypothesis (H) does not depart too far from reality.

Taking the geometric mean of the two estimates,

$$\theta' = \sqrt{2.46 \times 5.25} = 3.59 .$$

For this value of θ', the following theoretical values are found:

$$\lambda_T = R^{\infty}_{\theta',\omega} = e^{-\frac{\theta'}{\Theta}} = 0.455$$

$$1 - \chi_T = \frac{C^{\infty}_{\theta}}{C} = \left[1 - \frac{1 - e^{\theta' u}}{\Theta u}\right] e^{-\frac{\theta'}{\Theta}} = 0.82$$

(1) See Charts III A and III B below.

instead of the observed values $\mu = 0.32$ and $1 - \lambda = 0.90$ respectively.

Here again, if the exponential model is assumed to be correct, the divergences are the result of the incorrectness of hypothesis (H), but the concordance of orders of magnitude shows that the interpretation of reality corresponding to hypothesis (H) is acceptable as a first approximation.

As might have been expected, the values of $\theta' = 3.59$ which has been derived is of the same order of magnitude as the average production period, $\hat{\Theta} = 4.23$. In point of fact, if the supplementary hypothesis (H′) that all investments involve a production period equal in length to the average production period is made in addition to hypothesis (H), it follows necessarily that

$$\theta' = \hat{\Theta} = 4.23$$

It may be concluded from this that if the formulation of the exponential model can be considered as conforming with reality, approximate assessment of the real situation can be made by considering the hypotheses (H) and (H′) as valid.

Taking $\theta_2 = 3.59$, and using for illustrative purposes the value of $\theta_1 = 0.25$ (3 months) the following table can be derived.

TABLE 9

Characteristics for $\Theta = 4.56$ $i - \rho = 0.85\%$ $\rho = 1.7\%$ $\theta_1 = 0.25$ $\theta_2 = 3.59$ Exponential Hypothesis	Breakdown of R_C, R_ω, C, and C_ω			
	$0<\theta<0.25$	$0.25<\theta<3.59$	$0<\theta<3.39$	$3.59<\theta<\infty$
Share of consumed income $R^{\tau_2}_{\tau_1,C} / R_C$	0.051	0.479	0.531	0.470
Share of Primary income $R^{\tau_2}_{\tau_1,\omega} / R_\omega$	0.053	0.492	0.545	0.455
Share of Capital $C^{\tau_2}_{\tau_1,C} / C$	0.0014	0.179	0.180	0.820
Share of Primary capital $C^{\tau_2}_{\tau_1,\omega} / C_\omega$	0.0014	0.196	0.197	0.803
Capital Output Ratio $C^{\tau_2}_{\tau_1} / R^{\tau_2}_{\tau_1}$	0.125	1.704	1.554	7.956
Average amortisation period $\Theta^{\tau_2}_{\tau_1}$	0.124	1.718	1.562	8.15

6) *Whatever the amortisation schedule* $\varphi(\theta)$, for low values of $i - \rho$

(334-22) $$\Theta^{\tau_2}_{\tau_1} \sim C^{\tau_2}_{\tau_1} / R^{\tau_2}_{\tau_1}$$ [1]

It follows that if any two of the three quantities $C^{\tau_2}_{\tau_1}$, $R^{\tau_2}_{\tau_1}$ and $\Theta^{\tau_2}_{\tau_1}$ are known, the third can be estimated approximately.

(1) Relations (112-1) and (123-8).

If it is now assumed that the amortisation periods for equipment and structures are respectively 5 and 20 years [1], and if it is further assumed that the average production period for equipment goods is (Hypothesis H′)

$$\hat{\Theta} = 4.23$$

this leads to assessing at 10 and 25 years respectively the approximate length of the amortisation period of the primary inputs incorporated in equipment and structural assets.

If it is assumed that all investment consists of durable goods, and maintenance and repair outlays are abstracted, the position can be summarised as in Table 10 below (in $ billion) on the basis of the estimates in § 333-6 [2].

TABLE 10

	C	R_C	C/R_C	C in %	R_C in %
Inventories . . .	122.3	199.5	0.61	10.2	75.9
Equipment . . .	340.8	34.1	10	28.4	13.0
Structures . . .	736.6	29.5	25	61.4	11.1
Total . .	1199.7	263.1	4.56	100	100
Equipment and Structures . . .	1077.4	63.6	16.94	89.8	24.1

(1) These are the figures used by various fiscal administrative agencies.

(2) The values of R_C for equipment and structural assets are obtained by dividing the figures in the first column into those in the third. The R_C figures for inventories are then obtained by difference:

$$263.1 - 34.1 - 29.5 = 199.5 .$$

It can thus be seen that the figure of 10% of capital corresponds to 76% of consumed income for inventories, while for structural assets the corresponding figures are 61% for capital and 11% for consumed income. For equipment and structures as a whole 24% of consumed income corresponds to 90% of capital.

This table has an evident relationship with tables 7, 8 and 9.

The divergence between the amortisation period of some 17 years, found for equipment and structures considered as a whole, compared with values of some 8 and 10 years in tables 8 and 9 is due to the fact that the assumption has been made in the calculations in Table 10 that investment is constituted solely of durable goods, and that (given the non-availabilty of valuable data) repair and maintenance outlays included in the value of equipment and structures have not been taken into account. But the average amortisation period of repair and maintenance outlays is probably of the order of two to three years, and since their value is relatively high by comparison with total investment in new goods, it is probable that in a full calculation the gap would be considerably reduced, or might even disappear (1).

---

(1) Let C and R be the capital and consumed income corresponding to investment, C_1 and R_1 the share of that investment represented by new goods, and C_2 and R_2 the shares represented by repair and maintenance outlays (these symbols are valid only for the present note). Then

$$\frac{C}{R} = \frac{R_1}{R}\frac{C_1}{R_1} + \frac{R_2}{R}\frac{C_2}{R_2}$$

If the assumption is made that

$$\frac{R_1}{R} = \frac{1}{4} \qquad \frac{R_2}{R} = \frac{3}{4} \qquad \frac{C_1}{R_1} = 2.5 \qquad \frac{C_2}{R_2} = 17$$

then

$$\frac{C}{R} = 13.4$$

Thus, if annual repair and maintenance outlays account for 1/4 of the total annual resources devoted to investment, the average amortisation period falls from 17 to about 13 years. If it were assumed that repair and

Assuming that the expenditures required to maintain existing reproducible capital in a usable state amont to one-third of total annuel investment, and that the average amortisation period for these outlays is 2½ years, it is easy to arrive at the following table (¹).

TABLE II

	C	R_C	C/R_C	C in %	R_C in %
Inventories . . .	122.3	174.6	0.70	10.2	66.4
Repair and maintenance	73.0	29.2	2.5	6.1	11.1
Equipment (¹) . .	317.6	31.8	10	26.5	12.1
Structures (¹) . .	686 8	27.5	25	57.2	10.4
Total . .	1199.7	263.1	4.56	100	100
Equipment and Structures . . .	1077.4	88.4	12.18	89 8	33.6
Equipment and Structures (¹) . .	1004.4	59.2	16.96	83.7	22.5

(¹) Non-amortised value of initial installations, excluding the value of any repair and maintenancy charges incorporated.

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maintenance outlays represent 1/3 of total annual investment, the amortisation period would decline from 17 to 12 years approximately.

It can be seen from this calculation how much interest would attach to availability of estimates of the sums required to maintain reproduction capital in functioning order.

(¹) In allowing for repair and maintenance outlays, the elements of R_C which correspond to equipment and structural assets respectively in Table 10 become 34.1 $(1-x)$ and 29.5 $(1-x)$, so that the share of the value of fixed capital accounted for by repair and maintenance outlays becomes

$$[10 \times 34.1 + 25 \times 29.5]\, x \quad = 1077.4\, x\,.$$

The corresponding share of R_C is $1077.4\, x/\Theta'$, where Θ' is average amortisation period for repair and maintenance outlays. If K is the percentage

Obviously, given the non availability of valuable data, this can be no more than an indicative calculation. Nevertheless its results are quite suggestive if they are compared with the different items in Tables 7, 8 and 9 which relate to the percentage contribution of primary income to capital and consumed income.

Thus, the percentage 66.4 appearing in Table 11 in the breakdown of R_C differs relatively little from the 68.4 in Table 7, the 66.6 in Table 8 or the 54.5 in Table 9. Similarly, the overall percentage of 83.7 corresponding in the breakdown of capital to equipment and structures after exclusion of repair and maintenance expenditures is directly comparable to the figures of 70.8 and 82.0 in Table 8 and 9. Finally, tbe amortisation periods of 12.18 years for equipment and structural assets in Table 11 is directly comparable to the corresponding periods shown in Tables 8 and 9 as 9.56 and 8.15 respectively [1].

7) In usual practise, only those outlays whose amortisation period is one year or longer are in principle considered as investment.

However, in the first place, this limit is no more than a

---

which repair and maintenance outlays represent in annual investment, then we should have

$$\frac{1}{K} \times 1077.4\ x\ \frac{x}{\Theta'} = \frac{1}{1-K}(34.1 \times 29.5)\ (1-x)$$

whence

$$x = \frac{1}{1 + 16.94 \times \frac{1-K}{K}}$$

For $\Theta' = 2.5$ and $K = 1/3$,

$$x = 0.069$$

from which the various figures in Table II follow.

[1] If it is assumed that the cost of maintaining reproduction capital in operating condition represents one half of the sums devoted to investment, the breakdown percentages for C and R_C are found respectively as 10.2 - 11.5 - 24.8 and 53.6 for C and 59.0 - 20.9 - 11.3 and 9.7 for Rc.

principle. Often enough, an outlay which will to amortised over 18 months will still be charged against operating expenses. Let τ_1 denote the practical limit.

Secondly, investments whose amortisation period is below τ_1 and which are accounted as operating expenses contain both primary inputs and earlier inputs incorporated in the existing supplies and equipment. Thus, minor maintenance charges include at the same time wages, supplies produced some time earlier and the reward for the use of equipment. Thus the maintenance outlays which emerge into consumed income before τ_1 include not only the primary inputs supplied at moment t but also primary inputs corresponding to goods produced earlier. The primary inputs of moment t enter consumed income before τ_1 and at least as a first approximation, the second group of inputs can be considered as entering consumed income before $(\tau_1 + \hat{\Theta})$ where $\hat{\Theta}$ is the average length of the production period with the year as time unit [1].

It may be assumed, at least as a first approximation that the shares of the two categories of input are respectively proportional to

$$(334\text{-}23) \qquad \lambda' = \frac{R_\omega}{P^*} \qquad 1 - \lambda' = 1 - \frac{R_\omega}{P^*} \qquad [2]$$

If we consider the weighted average

$$(334\text{-}24) \qquad \theta' = \lambda'\tau_1 + (1 - \lambda')(\tau_1 + \hat{\Theta}) = \tau_1 + (1 - \lambda')\hat{\Theta}$$

[1] $\hat{\Theta}$ is defined by relation (112-2).
[2] Relation (333-5).

it appears possible to consider as a first approximation that investment goods, understood in the usual meaning of the term, include all primary inputs which emerge in consumed income after θ' .

Since

$$\theta' = 3.59 \qquad \hat{\Theta} = 4.23 \qquad \lambda' = 0.654 \qquad (^1)$$

we find

$$\tau_1 = 3.59 - (1 - 0.657) \times 4.23 = 2.14$$

a value which appears to be quite reasonable in the light of the discussion above.

8) It is also possible to derive an approximate representation of reality by substituting the following hypothesis (H″) for the two hypotheses (H) and (H′): the amortisation schedule for the primary income embodied in stocks and goods in course of production follows an exponential law:

$$(334\text{-}25) \qquad \varphi_1(\theta) = \frac{e^{-\frac{\theta}{\Theta_1}}}{\Theta_1}$$

(1) Table 7.

Thus, the share of primary inputs which have been incorporated in stocks and goods in course of production and emerge in consumed income between θ and $\theta + d\theta$ is

$$(334\text{-}26) \qquad (1-\lambda)\, R_\omega \frac{e^{-\frac{\theta}{\Theta_1}}}{\Theta_1}\, d\theta$$

while the share of primary inputs which have been incorporated in equipment and structures and emerge in consumed income between θ and $\theta + d\theta$ is

$$(334\text{-}27) \qquad R_\omega \left[\frac{e^{-\frac{\theta}{\Theta}}}{\Theta} - (1-\lambda) \frac{e^{-\frac{\theta}{\Theta_1}}}{\Theta_1} \right] d\theta$$

For the first of these two shares (stocks and goods in course of production), the general formulae of the exponential model may be applied by substituting Θ by Θ_1, so that the capital corresponding to expression (334-25) is (1)

$$(334\text{-}28) \qquad C_1 = \frac{\Theta_1}{1-(i-\rho)\Theta_1}\,(1-\lambda)R_\omega$$

and, in the notation of § 333

$$(334\text{-}29) \qquad \chi = \frac{C_1}{C} = \frac{1-\lambda}{1-(i-\rho)\Theta_1}\, \frac{\Theta_1}{\Theta_0}$$

(1) Relations (251-5) and (251-9).

so that

$$\Theta_1 = \frac{\Theta_0}{\frac{(1-\lambda)}{\chi} + (i-\rho)\,\Theta_0} \qquad (334\text{-}30)$$

The following values have been found earlier:

$\lambda = 0.316 \qquad \chi = 0.10 \qquad i - \rho = 0.85\% \qquad \Theta_0 = 4.74$

whence

$$\Theta_1 = 0.69$$

Thus the value of Θ_1 found here is fully comparable with the values of 0.61 and 0.71 in tables 10 and 11.

For the value of $\theta' = 3.59$ found earlier.

$$e^{-\frac{\theta'}{\Theta_1}} = .0055$$

indicating that for $\theta = \theta'$, the primary income corresponding to stocks and goods in course of production is almost completely amortised .

This calculation shows that the hypothesis (H″) does not differ greatly from the hypothesis (H).

If Θ_1 is put equal to 0.69, the average amortisation period for investment, Θ_2, is

$$(334\text{-}31) \qquad \Theta_2 = \frac{\int_0^\infty \theta \left[\frac{e^{-\frac{\theta}{\Theta}}}{\Theta} - (1-\lambda)\frac{e^{-\frac{\theta}{\Theta_1}}}{\Theta_1}\right] d\theta}{\lambda}$$

$$= \frac{\Theta - (1-\lambda)\Theta_1}{\lambda}$$

$$= \frac{4.56 - 0.684 \times 0.69}{0.316}$$

whence

$$\Theta_2 = 12.9$$

This value must be compared with the values shown in tables 8, 9 and 11 of 9.56, 8.15, and 12.18.

9) Taking hypothesis (H) in conjunction with table 9, the average amortisation period of the primary inputs corresponding to investment I is 8.15, whereas it is 12.9 in the calculation immediately above. At first sight, these figures appear to be rather low for the entire range of equipment and structure (1).

(1) At all events, it should be borne in mind that, *independently of the basic assumptions of the exponential model,* we should have (condition 123-12)

$$\Theta \leq \frac{\gamma_c}{1-\gamma_c\ (i-\rho)} \qquad \text{for} \qquad i-\rho \geq 0$$

a relation which holds for primary inputs taken as a whole whatever the function $\varphi(\theta)$.

This condition depends only on the assumption of invariance over time

It is quite easy to make a direct estimate of a reasonable order of magnitude. From § 333, the value of equipment and structures (in $ billion) is

$$C_1 = (1 - \chi)\, C = 1199.7 - 122.3 = 1077.4 \,.$$

The annual amortisation of this capital is

$$\mathfrak{A}_1 = I - (1 - \chi) \frac{dC}{dt}$$

$$= I - (1 - \chi)\, \rho\, C$$

i.e. from the estimate derived earlier

$$\mathfrak{A}_1 = 121.8 - 0.9 \times 0.017 \times 1077.4$$

$$= 121.3 - 16.5 = 105.3$$

Thus, a first approximation to the length of the amortisation period for investments is derived as

$$\Theta_i = 1077.4 / 105.3 = 10.23 \qquad (^1)$$

of the function $\varphi(\theta)$ i.e. the invariance over time of the composition of the labor force. Thus, for the U.S.A. in 1956, we should have

$$\frac{4.56}{1 - 0.0085 \times 4.56} = 4.74 \,.$$

Thus there are *structural reasons* for the average amortisation period *for the whole range* of primary inputs to remain relatively low.

(1) It will be recalled that for fairly low values of ρ and $i - \rho$, the average amortisation period of an element of capital is approximately equal to the ratio of its value to the annual amount of amortisation, whatever the amortisation schedule.

However, to obtain the average amortisation period for the corresponding primary inputs, this figure must be increased by the average length of the production period for investment goods, whose order of magnitude is $\theta' = 3.59$ (1). The average duration of the amortisation period for the primary inputs corresponding to investment is then

$$\Theta_2 = 10.2 + 3.59 = 13.79$$

which may be compared with the estimates of 8.15 in Table 9 and 12.9 in § 334-8. From this point of view, the approximative hypothesis (H″), which leads to a value of 12.0 (2) is to be preferred to the interpretation underlying hypothesis (H), which leads to a value of 8.15.

Clearly, if it is assumed that the general formulation of the exponential model is valid, the difference between the theoretical value of 12.9 calculated under hypothesis (H″) and the observed value of 13.8 may be explained by the inaccuracy of the assumption in (H″). But in the light of the appealing nature of this assumption, at least as a first approximation, the concordance of the orders of magnitude of the two estimates can be considered as underscoring the validity of the exponential model. At the same time, the results derived using interpretation (H) show that this hypothesis offers only a roughest resemblance to reality.

10) Since the average production period for consumed income is $\hat{\Theta}$, it may be assumed as a first approximation that the average production period for investment goods is also $\hat{\Theta}$.

It is then also possible to represent as a first approximation the reality by the following model (Hypothesis H‴). In each

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(1) § 334-5.
(2) § 334-8.

period of time equal to $\hat{\Theta}$, total production is equal to consumed income plus invested income, and the corresponding investment is completely used up during the following period. In the first period, the share of $R_\omega(t)$ which appears in consumed income is $(1 - \lambda)$, while λ is invested. In the second period, the share of primary income $R_\omega(t)$ which appears in consumed income is $\lambda(1 - \lambda)$ while λ^2 is invested. In the n^{th} period, the fraction of primary income $R_\omega(t)$ which appears in consumed income R_c^n of that period is given by (1)

$$\lambda^{n-1} (1 - \lambda) .$$

Thus, as a result of this process, the amortisation schedule of primary income is of exponential form, and that over a period of length $\hat{\Theta}$, the amount of non-amortised primary income is reduced in the ratio of 1 to λ. It follows that if the formulation of the exponential model is taken as exact, the share λ of primary income incorporated in investment is

$$\lambda = e^{-\frac{\hat{\Theta}}{\Theta}}$$

since Θ is the lengh of the amortisation period.

For

$$\hat{\Theta} = 4.23 \qquad\qquad \Theta = 4.56$$

(1) Naturally we have

$$1 = (1 - \lambda) \ (1 + \lambda + \ldots + \lambda^n + \ldots) .$$

the value of λ is found as

$$\lambda = 0.40 \; .$$

This figure is of the same order of magnitude as the 0.316 found in § 333-6. Here again, the divergence can be attributed to the approximative nature of the hypothesis (H‴), on the assumption that the exponential model is valid. But if this approximation is admitted, the concordance of orders of magnitude indicates that, if hypothesis (H‴) is taken to be approximately true, the exponential model can be considered as not departing very far from reality.

This reasoning shows, at least intuitively, how the assumption of exponential amortization of primary income is justified. If the general formulation of the exponential model is taken as exact, it also supplies an estimate of the order of magnitude of the proportion λ of primary income incorporated in investment, which agrees with observed data (1).

(1) Of course it would be of the greatest possible interest to generalise the above reasoning to the case in which the lengths of the production and amortisation periods of different types of investment are no longer considered as uniform, but the frequency of a production period extending from θ to $\theta + d\theta$ would be $\hat{\varphi}(\theta)d\theta$, and the frequency of an amortisation period for primary inputs extending from θ to $\theta + d\theta$ would be $\varphi(\theta)d\theta$, with the discrete model becoming continuous (See § 333-2 above). Up to now, I have been unsuccessful in making a calculation of this kind. If it could be done, it would doubtless enable the validity of the exponential model to be verified efficiently, by determining the relations between the quantities λ, χ and Θ and the data R_0, C et ρ, using this model.

In present circumstances, and in the absence of a calculation of this kind progress can only be made on the basis of rough estimates.

Turning now to the overall amortization period for primary income, here this period, given the assumptions made, is equal to

$$(334\text{-}32) \qquad \Theta = (1-\lambda)\frac{\hat{\Theta}}{2} + \lambda(1-\lambda)\left(\hat{\Theta} + \frac{\hat{\Theta}}{2}\right) + \dots + \lambda^{n-1}(1-\lambda)\left[(n-1)\hat{\Theta} + \frac{\hat{\Theta}}{2}\right]$$

$$= \frac{\hat{\Theta}}{2} + \lambda(1-\lambda)[1 + 2\lambda + \dots + (n-1)\lambda^{n-2} + \dots]\hat{\Theta}$$

$$= \left[\frac{1}{2} + \lambda(1-\lambda)\frac{d}{d\lambda}\left(\frac{1}{1-\lambda}\right)\right]\hat{\Theta}$$

whence

$$(334\text{-}33) \qquad \frac{\Theta}{\hat{\Theta}} = \frac{1}{2}\frac{\lambda}{1-\lambda}$$

For $\hat{\Theta} = 4.23$, $\lambda = 0.316$, (1) we find

$$\Theta = 4.05 .$$

This value is of the same order of magnitude as the values of Θ and $\hat{\Theta}$, 4.56 and 4.23 respectively, found earlier (2).

11) *Of course, the concordance of the orders of magnitude of the different values of the quantities studied in the preceding calculations cannot be considered as proof of the validity of the exponential model. The successive hypotheses*

(1) § 333-6 and 334-1.
(2) § 334-1.

(H), (H'), (H'') and (H''') cannot be considered as rigourously valid, and the computations have involved a certain number of further approximations. At the same time, the non rigorous validity of hypotheses (H) to (H''') may account for the divergences noted between the various estimates that have been derived.

Be this as it may, the rather striking coherence in the whole set of results derived using these different assumptions and applying quite rough calculating techniques, does seem to show, given the apparently reasonable nature of the interpretations (H), (H'), (H'') and (H''') which have been succesively considered, that the exponential hypothesis can be considered as a fairly close reflection of reality.

12) Finally, the results of the analysis described above are not out of line with data based on the composition of the labor force.

a) Given the composition of the labor force, the estimate derived from the above analysis that the share of primary income devoted to investment is of the order of 30% to 35% seems quite reasonable. Thus, in 1955, the composition by sector of the working population in the U.S.A. was [1]:

Agriculture	11.8%
Industry	29.2%
Mining	1.4%
Construction	4.9%
Transport	7.2%
Trade	23.0%
Services	10.4%
Government	12.1%
Total	100 %

[1] Statistical Abstract, 1959, pp. 206 and 210. A detailed discussion of these figures is beyond the scope of this paper.

About one-half of industrial production consists of durable goods (1), and it can be taken that about ¼ of the output of non-durables is reinvested. Thus, it can be considered that 18.5% of the labor produces primary inputs for durable goods. It may also be considered that the construction sector's activity results for the greatest part in the production of durable goods, while the corresponding figure for mining may be of the order of 50%, so that the corresponding percentage of the labor force for the two sectors is about 5%. The contribution of inputs from the trade sector to the production of durable goods can be estimated at some 20% of its output, accounting for a further 4½% of the labor force. In total, the share of primary inputs in durables production arising in the industry, mines, construction and trade sectors adds up to a total of some 28% of primary income.

It may not be unreasonable to assume that activity in the transport and governmental sectors (19.3%) is distributed between durables and non-durables in line with the average for other sectors, so that the total contribution to production of non-durables of sectors other than transport and government is

$$100 - 28 - 19.3 = 52.7\% .$$

It follows from this set of estimates taken as a whole that the share of primary income going to non durables and investment may he respectively of the order of

$$\frac{100}{100 - 19.3} \cdot 52.7 = 65\%$$

$$\frac{100}{100 - 19.3} \cdot 28 = 35\% .$$

(1) ALLAIS, *Tokyo Memorandum* (1960 A), p. 32.

Thus, we find for the percentage of the primary income which is embodied in investment a figure which is of the same order of magnitude as the figure which is given by the preceding analysis.

b) From Tables 8 and 9, some 5% and 20% of primary income emerge in consumed income during the following quarter and year. These estimates seem also perfectly compatible with the figures for the composition of the labor force given above.

But it should be stressed that analyses based on consideration of the breakdown of the labour force, given the present state of the analysis must be considered as very rough, and of indicative value only.

Determination of the Function $\varphi(\theta)$

335. It does not appear to be absolutely impossible to determine the function $\varphi(\theta)$ from an analysis of the composition of the working population and available data on the structure of production, at least on an approximative basis.

Unfortunately, this calculation involves a great deal of work, and up to now I have not been able to find the time to do it, even though the results of such an analysis would clearly be of the greatest interest.

It may well be, and in the author's view it is quite likely, that over the useful range of variation of θ i.e. from 0 to 100 years, the exponential form

$$\varphi(\theta) = \frac{1}{\Theta} e^{-\frac{\theta}{\Theta}} \tag{335-1}$$

provides a good representation of the function which would be derived empirically, at least as a first approximation.

Were it to appear that an exponential expression cannot represent the function adequately, the assumption could be made that the function $e^{\mu\theta}\varphi(\theta)$, where μ is an appropriate constant, could be developed as a Taylor series and quite well represented by its first terms. This is a relatively weak hypothesis, and one which does not appear to be inacceptable. The general properties arising when this hypothesis is postulated are treated in the appendix [1].

As an illustration, values of

$$\left\{\begin{array}{ll}
(335\text{-}2) & \beta(\theta) = \dfrac{e^{-\frac{\theta}{\Theta_0}}}{\Theta_0} \\[2ex]
(335\text{-}3) & \varphi(\theta) = \dfrac{e^{-\frac{\theta}{\Theta}}}{\Theta} \\[2ex]
(335\text{-}4) & e^{-\frac{\theta}{\Theta}} \\[2ex]
(335\text{-}5) & 1 - e^{-\frac{\theta}{\Theta}} \\[2ex]
(335\text{-}6) & \dfrac{C_\theta^\infty}{C} = \left[1 - \dfrac{1 - e^{\theta u}}{\Theta u}\right] e^{-\frac{\theta}{\Theta}}
\end{array}\right.$$

(1) The study contained in the appendix considers the function $\beta(\theta)$ from zero to infinity; but the characteristics at infinity of the functions $\beta(\theta)$ and $\varphi(\theta)$ need not be taken into account in numerical applications. It is therefore possible to consider only a finite range of variation of θ, for example the interval (0, 100).

for

(335-7) $\Theta_0 = 4$ years $\Theta = 3.5$ years $u = 4\%$

are given in the following table [1] [2].

The last three expressions listed represent respectively the amortized and non-amortized components of primary income and the share of capital corresponding to primary inputs which are amortized after an interval of at least θ [3].

TABLE 12

θ years	$\beta(\theta)$	$\varphi(\theta)$	$e^{-\frac{\theta}{\Theta}}$	$1-e^{-\frac{\theta}{\Theta}}$	C_θ^∞ / C
0	0.250	0.285	1.	0	1
1/12	0.244	0.279	0.976	0.023	0.999
1/4	0.234	0.266	0.931	0.068	0.997
1/2	0.220	0.247	0.866	0.133	0.991
1	0.194	0.214	0.751	0.248	0.970
2	0.151	0.161	0.564	0.425	0.900
3	0.118	0.121	0.424	0.575	0.810
4	0.092	0.091	0.318	0.681	0.714
5	0.071	0.068	0.239	0.760	0.618
10	0.020	0.016	0.057	0.942	0.259
15	0.005	0.003	0.013	0.986	0.094
20	0.001	0.0009	0.003	0.996	0.032
25	0.2×10^{-3}	0.2×10^{-3}	0.8×10^{-3}	0.999	0.010
50	0.1×10^{-5}	0.2×10^{-6}	0.6×10^{-6}	1	0.3×10^{-4}
100	0.2×10^{-11}	0.1×10^{-12}	0.4×10^{-12}	1	0.1×10^{-9}

(1) The relations (335-2) to (335-6) are deduced from (250-5) ($k = 1$), (334-1), (334-5), (334-3) and (334-8). The relation (335-6) is deduced from (334-8), (251-5) and (251-9).

(2) From (251-5) we have

$$u = i - \rho = \frac{1}{\Theta} - \frac{1}{\Theta_0}$$

for $\Theta_0 = 4$, $\Theta = 3.5$ we derive $u = 3.55\%$.

The values of Θ_0 and Θ considered here correspond to the average estimates there of (§ 321 and 326), and differ from those considered in § 334.

(3) Three significant places have been kept.

The various results corresponding to different values of Θ, Θ_o and $u=\rho$ are given in charts I, II, III A and III B.

The Exponential Hypothesis and Reality

336. The exponential hypothesis appears to be a very restrictive one, at least at first sight.

However, it is possible to verify that it may in fact provide a quite fair approximation to real conditions.

From this point of view, there are three particularly important coefficients to be considered. These are k, Θ_o and Δ defined by the relations (1)

$$k = \int_o^\infty \beta(\theta) d\theta \tag{336-1}$$

$$\Theta_o = \frac{1}{k} \int_o^\infty \theta \beta(\theta) d\theta \tag{336-2}$$

$$\Delta = \frac{1}{2k\Theta_o^2} \int_o^\infty \theta^2 \beta(\theta) d\theta \tag{336-3}$$

As it has already been noted, k should be taken as not greatly differing from unity (2).

(1) Relations (211-3), (220-5) and (220-6).
(2) § 311.

CHART I

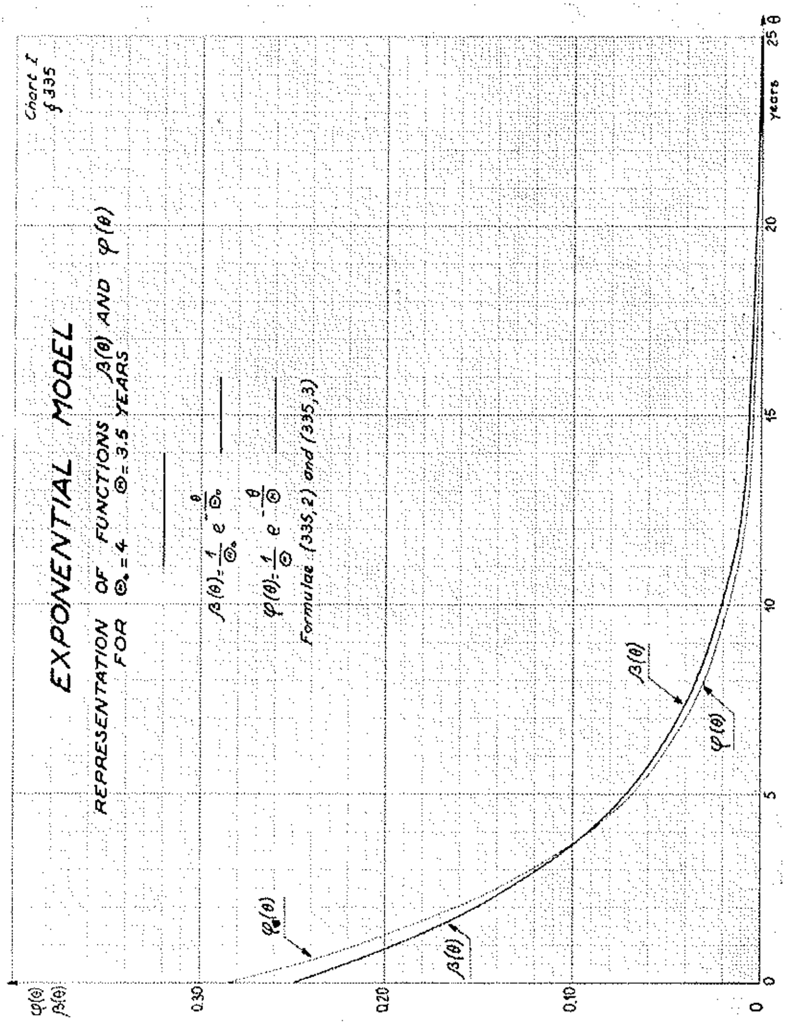
EXPONENTIAL MODEL
REPRESENTATION OF FUNCTIONS β(θ) AND φ(θ)
FOR Θ₀ = 4 Θ = 3.5 YEARS
β(θ) = 1/Θ₀ e^(−θ/Θ₀)
φ(θ) = 1/Θ e^(−θ/Θ)
Formulae (335,2) and (335,3)
Chart I
§ 335
φ(θ)
β(θ)
0.30
0.20
0.10
0
5
10
15
20
25
θ
years

CHART II

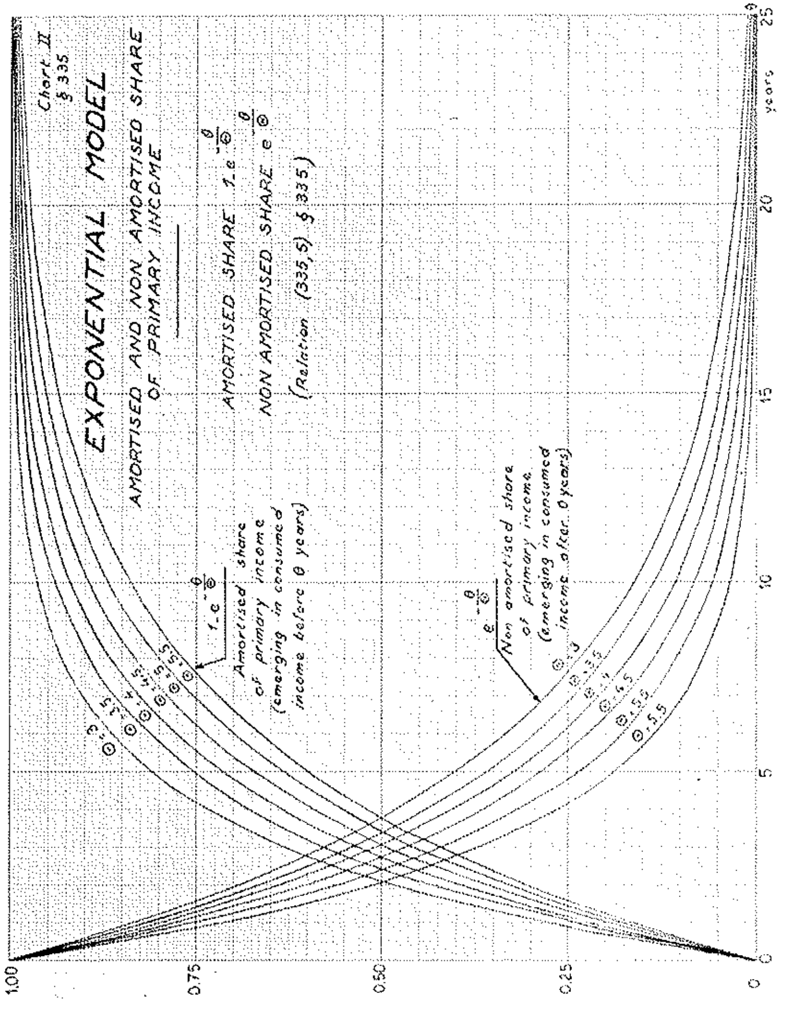
Chart II
§ 335
EXPONENTIAL MODEL
AMORTISED AND NON AMORTISED SHARE
OF PRIMARY INCOME
AMORTISED SHARE 1-e^-θ/Θ
NON AMORTISED SHARE e^-θ/Θ
(Relation (335,5) § 335)
1-e^-θ/Θ
Amortised share of primary income (emerging in consumed income before θ years)
e^-θ/Θ
Non amortised share of primary income (emerging in consumed income after θ years)
1.00
0.75
0.50
0.25
0
0
5
10
15
20
25
years

CHART III A

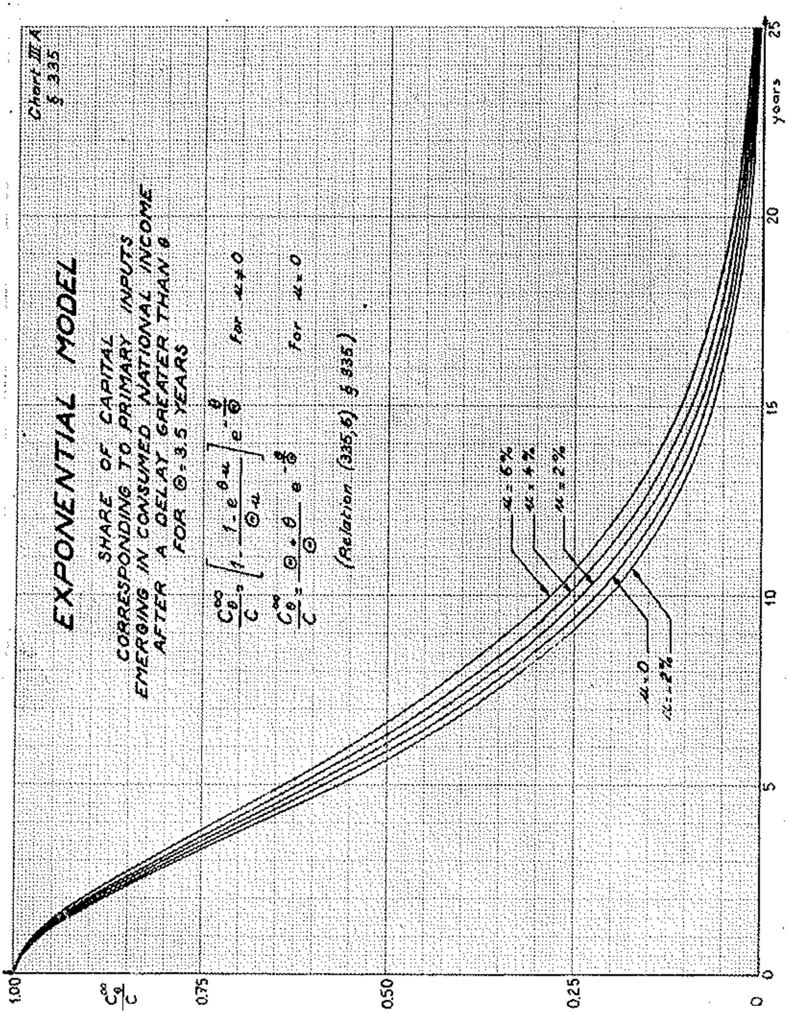
Chart III A
§ 335
EXPONENTIAL MODEL
SHARE OF CAPITAL
CORRESPONDING TO PRIMARY INPUTS
EMERGING IN CONSUMED NATIONAL INCOME
AFTER A DELAY GREATER THAN θ
FOR Θ = 3.5 YEARS
for u ≠ 0
for u = 0
(Relation (335,6) § 335)
u = 6%
u = 4%
u = 2%
u = 0
u = -2%
years
0
5
10
15
20
25
1.00
0.75
0.50
0.25
0

CHART III B

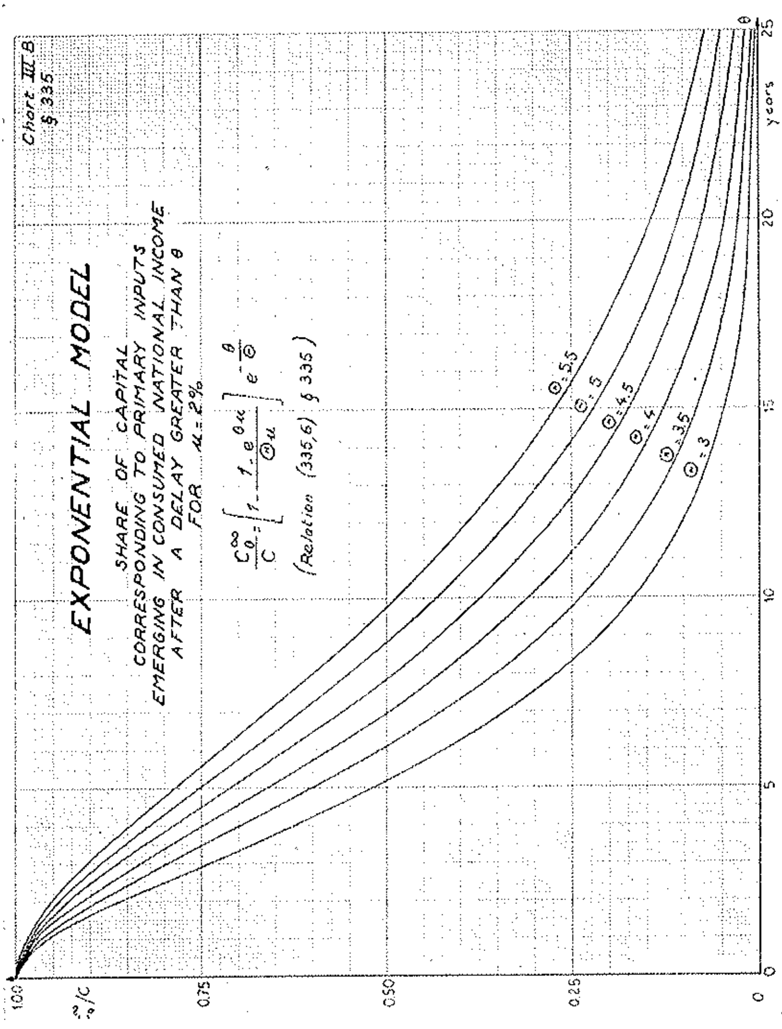
Chart III B
§ 335
EXPONENTIAL MODEL
SHARE OF CAPITAL
CORRESPONDING TO PRIMARY INPUTS
EMERGING IN CONSUMED NATIONAL INCOME
AFTER A DELAY GREATER THAN θ
FOR u = 2%
$\frac{C_0^\infty}{C} = \left[1 - \frac{1 - e^{\theta u}}{\Theta u}\right] e^{-\frac{\theta}{\Theta}}$
(Relation (335,6) § 335)
Θ = 5.5
Θ = 5
Θ = 4.5
Θ = 4
Θ = 3.5
Θ = 3
1.00
0.75
0.50
0.25
0
5
10
15
20
25
years
θ

Since we have

$$(336\text{-}4) \qquad \gamma_c = \Theta_0 \text{ for } i - \rho = 0 \qquad ([1])$$

and $i - \rho$ is in general small, it follows that when an estimate of γ_c is available, it is possible to derive an order of magnitude of Θ_o.

Then if we consider all those models for which

$$(336\text{-}5) \qquad k = 1 \qquad \Theta_0 \sim \gamma_c$$

they differ from each other by the order of magnitude of the parameter Δ, *which is equal to unity in the exponential model.*

Since we have

$$(336\text{-}6) \qquad \varphi(\theta) = \frac{\beta(\theta) e^{-(i-\rho)\theta}}{k\,\psi(i-\rho)} \qquad ([2])$$

and

$$\psi(0) = 1 \qquad ([3])$$

it follows that as a first approximation, $\varphi(\theta)$ differs relatively little from $\beta(\theta)$ for $k = 1$.

It then follows that, *at least as a first approximation,* Δ *is greater than or smaller than unity according to whether the amortization is more or less rapid than that corresponding to*

([1]) Relation (240-10).
([2]) Relation (223-7).
([3]) Relation (220-2).

the exponential amortization which is a characteristic of the exponential model (1).

It is shown in the Appendix to the present study that under the very general hypothesis that the function $\beta(\theta)e^{\mu\theta}$ can be developed as a Taylor series, at least for a certain value of μ, the coefficient Δ (which is equal to 1 in the exponential model) will not differ generally from unity by more than ± 0.5, for hypotheses which can reasonably be accepted.

It follows from this that while the exponential model clearly may not offer a perfect portrayal of real conditions, it doubtless does not differ therefrom all that much, so that, at least in terms of a first approximation, it is capable of depicting the essential features of reality quite correctly.

If this approximation is deemed insufficient, the development of the function $\beta(\theta)e^{\mu\theta}$ as a Taylor series of which the very first terms only are retained, provides as far as can be judged a reasonable representation of actual conditions (2).

The Value of the Model

337. It follows from the preceding dimission that:

a) the model is consistent with the information which can be obtained from available statistical data;

b) it cannot be proved that the assumptions made are the only ones which would give results consistent with available data.

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(1) From relation (220-6),

$$\Delta \sim \frac{1}{2k\Theta_0}\int_0^\infty \theta^2 \varphi(\theta)\, d\theta$$

Thus if $\varphi(\theta)$ declines more rapidly than $e^{-\frac{\theta}{\Theta}}/\Theta$ we have $\Delta < 1$ since $\Delta = 1$ for $\varphi(\theta) = e^{-\frac{\theta}{\Theta}}/\Theta$.

(2) See Appendix.

Clearly, there exists an infinity of different mathematical functions which in a restricted field give analogous results, and *this characteristic simply derives from the nature of things.*

The point is that everything takes place as if the theory presented and the model which illustrates that theory were correct. As far as I know no other theory, no other model, have ever been put forward in order to explain the same facts.

It may therefore reasonably be suggested that the theory and the model presented here can be accepted and used, at least as working tools, so long as no alternative theory or model is advanced which leads to results which concord still better with the real situation.

In any event, the theory and the model which have been discussed here have the advantages that they represent analytical tools and that they oblige the economist to reflect on a large numher of issues which hitherto have been insufficiently studied, not to say completely neglected (1).

Finally, it may be observed that every theory has a twin aim; on the one hand to describe and explain reality (2), but at the same time to constitute a guide to efficient action. For a given degree of approximation, the best theory at any given moment is the one which fulfils the condition of being the most convenient, or in other words, the simplest of all those which represent reality with that degree of approximation. If this criterion be admitted, and personally I know of no other (3), the theory given here has the double advantage of being on the one hand very simple, but also of describing and explaining reality, so far as it can be comprehended with the information presently available to us, i.e. of being compatible with observed facts and simultaneously establishing coherent and simple links between these facts.

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(1) Doubtless precisely because they were too difficult.

(2) In other words, to find relationships between the different aspects of reality, expressing the most complex in terms of the most simple.

(3) See for instance HENRI POINCARÉ, *The Value of Science* (Flammarion).

Part IV

APPLICATIONS

400. The preceding theory admits of several particularly suggestive applications.

For simplicity's sake, the following exposition will be developed in terms of the exponential model with first order homogeneity.

A. — THE POSSIBILITY OF INCREASING REAL PER CAPITA NATIONAL INCOME BY INCREASING CAPITAL INTENSITY

410. From relation (251-15) we have for $k=1$

(410-1) $$\frac{\overline{R}_C}{\overline{R}_{CM}} = \frac{\Theta_0}{\Theta} e^{1-\frac{\Theta_0}{\Theta}} = \left[1+\Theta_0(i-\rho)\right] e^{-\Theta_0(i-\rho)}$$

and from (251-11) and (210-10)

(410-2) $$\Theta = \gamma_C = C/R_C$$
$$\Theta_0 = \gamma_{C,0}$$

All the statistical data analysed lead to the conclusion that in present conditions Θ_0 is of the order of magnitude

$$\Theta_0 = 4$$

the time unit used being the year, and we have probably (1)

$$3.8 < \Theta < 4.6 .$$

The increase in real consumable income $\bar{R}_C$ which can be had by moving from a value Θ to the value Θ_0 corresponding to the maximum value of $\bar{R}_C$ is equal to

(410-3) $$g = \frac{\bar{R}_{CM} - \bar{R}_C}{\bar{R}_C}$$

or, from relation (410-1)

(410-4) $$g = \frac{\Theta}{\Theta_0} e^{\frac{\Theta_0}{\Theta} - 1} - 1$$

In each situation, the value of the difference $i - \rho$ is given by the relation (251-5)

(410-5) $$i - \rho = \frac{1}{\Theta} - \frac{1}{\Theta_0}$$

(1) This question cannot be taken up here.

From the foregoing we derive the following table of values of g and i (in %)

TABLE 13

Values of g in %								
Θ_0	3.8		4		4.2		4.4	
$\Theta = \gamma_c$	g	$i-\rho$	g	$i-\rho$	g	$i-\rho$	g	$i-\rho$
4.4	1.0	−3.6	0.4	−2.3	0.1	−1.1	0	0
4.2	0.5	−2.5	0.1	−1.2	0	0	0.1	1.1
4	0.1	−1.3	0	0	0.1	1.2	0.5	2.3
3.8	0	0	0.1	1.3	0.5	2.5	1.1	3.6
3.6	0.2	1.5	0.6	2.8	1.3	4.0	2.2	5.1
3.4	0.6	3.1	1.4	4.4	2.4	5.6	3.7	6.7
3.2	1.6	4.9	2.7	6.3	4.2	7.4	5.8	8.5
3.0	3.1	7.0	4.7	8.3	6.6	9.5	8.7	10.6
2.5	10.7	13.7	13.9	15.0	17.5	16.2	21.5	17.3
2.0	29.5	23.7	35.9	25.0	43.1	26.2	50.9	27.3
1.5	82.9	40.4	98.6	41.7	116.0	47.9	135.8	43.9
1.0	332.8	73.7	402.1	75.0	484.1	76.2	581.1	77.3
0.5	957.4	173.7	1360.8	175.0	1938.5	176.2	2762.5	177.3

The two charts IV and V which follow give the values of $\bar{R}_C/\bar{R}_{CM}$ as functions of $i-\rho$ and γ_C.

Real consumable national income is a maximum for $\Theta = \Theta_0$.

We see that in an initial situation such as that of the United States in 1956, in which the capital-output ratio γ_c is of the order of 3.5, the maximum is very nearly attained, and that for a corresponding value of Θ_0 of 4, the possible increase is only of the order of 1%.

The same is true for most western economies. The levels of the capital-output ratio are such that they can be considered as corresponding to a situation which is very near to the capitalistic optimum.

Even in the case of economies in which the capital-output ratio is only of the order of 2 (if there are any such), the increase in real income to be derived from a lengthening of

CHART IV

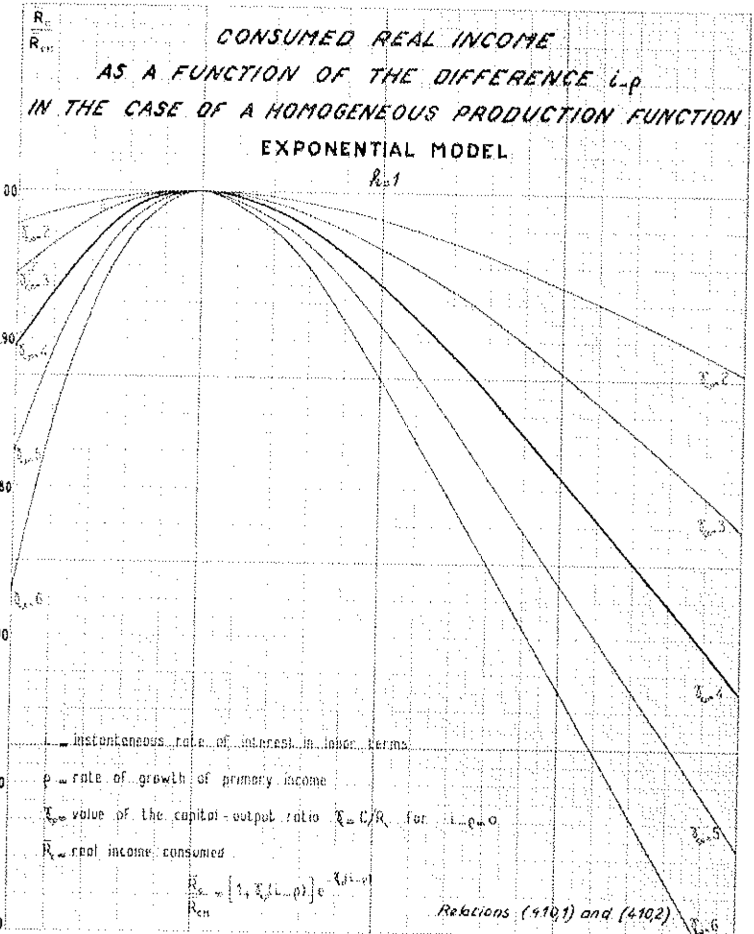
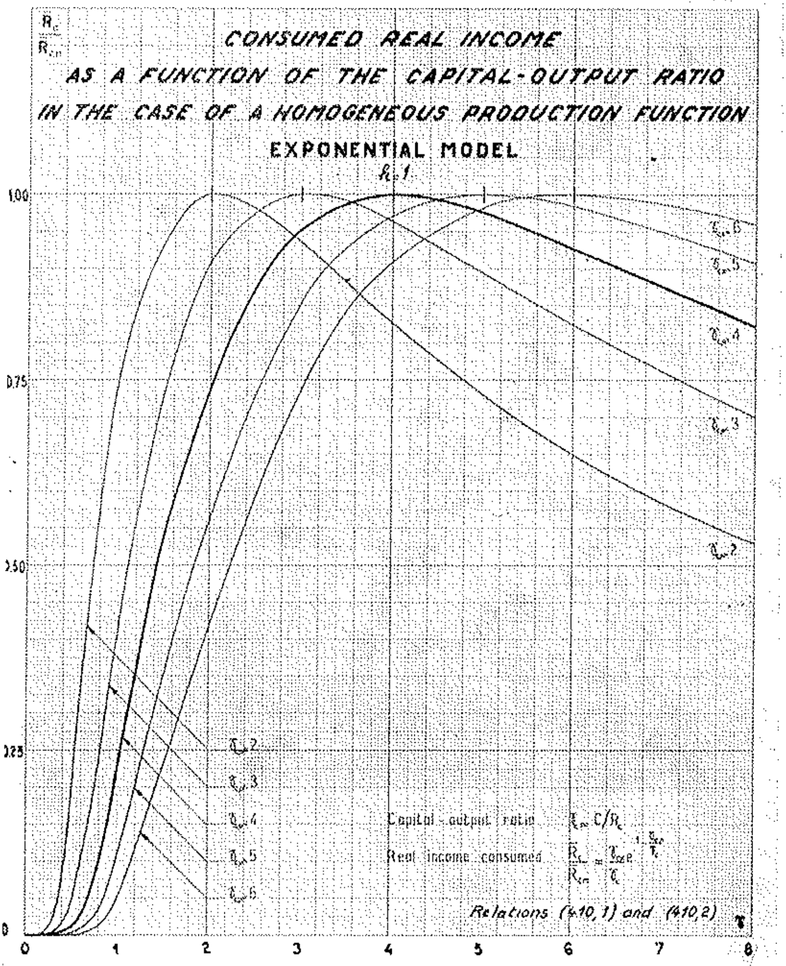

CONSUMED REAL INCOME
AS A FUNCTION OF THE DIFFERENCE i-ρ
IN THE CASE OF A HOMOGENEOUS PRODUCTION FUNCTION
EXPONENTIAL MODEL
i — instantaneous rate of interest in labor terms
ρ — rate of growth of primary income
value of the capital-output ratio C/R for i-ρ=0
real income consumed
Relations (410,1) and (410,2)
-10%
0
10%
20%
i-ρ
30%
.90
.80
.70
.60
.50

CHART V


CONSUMED REAL INCOME
AS A FUNCTION OF THE CAPITAL-OUTPUT RATIO
IN THE CASE OF A HOMOGENEOUS PRODUCTION FUNCTION
EXPONENTIAL MODEL
Capital-output ratio
Real income consumed
Relations (410,1) and (410,2)
1.00
0.75
0.50
0.25
0
0
1
2
3
4
5
6
7
8

the period of production, in the case where $\Theta_0 = 4$, is only of the order of 36%, the difference $i - \rho$ being some 25%.

It is only for capital-output ratios below 1.5, i.e. very low ones indeed, that gains of real income could be very appreciable: but in any case for which estimates has been available no such low values of γ_C have been met.

Clearly it follows that:

1) *developed economies have relatively little to gain from an increase in savings;*

2) *that less developed countries, on the contrary, can realise substantial gains from greater savings efforts when the marginal productivity of capital is very high, but that these gains are however not as large as is generally believed.*

Significance of the Results Obtained

411. It can thus be seen that the real income gains which can be had by lengthening the average duration of the capitalistic process are in general overestimated, not to say *considerably* overestimated.

But a basic assumption of the model should be borne in mind when making this interpretation. This is that there is optimum management in all the situations considered, implying that existing savings are utilised in the best possible way. (Paretian optimum hypothesis) (1).

This being so, although the results derived above show that greater savings would not be very advantageous in an economy in which there is optimum management, they in no way imply that increased savings would not be very productive if savings already accumulated had in the past been invested in mistaken directions.

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(1) § 117.

In this latter case, a growth in new savings could have major beneficial effects. *They would however be due not to the inadequate volume of savings which had been accumulated in the past, but to their defective utilisation in the form of economically erroneous investments.*

In a situation of this kind, it is not the global volume of accumulated savings which is at issue, but its use in the form of non-productive investments. Thus inefficient management rather than insufficient capital intensity must be considered as providing the explanation of unduly low average productivity (1).

It is thus clear that the important factor is not so much the volume of accumulated savings evaluated in terms of the number of years national income it represents — which does not vary greatly from one country to another, even including underdeveloped countries — but *the use that these savings are put to.*

(1) See my book: *A United Europe, the Road to Prosperity* (CALMANN-LÉVY, 1960), Chap. II, pp. 28 to 38 and 42 to 51 and Appendix C (295 to 297).

B. — PROCESS OF MAXIMUM GROWTH OF PRODUCTION PER UNIT OF PRIMARY INCOME FOR GIVEN TECHNICAL KNOWLEDGE

Decrease in Real Consumed Income due to Growth in Primary Income

420. We have seen [relation (233-2)] that in the case of the general model we have

(420-1) $$\overline{R}_{CM} = K(t)\, e^{-k\Theta_0 \rho}$$

where $\overline{R}_{CM}$ is the maximum value of the real national consumed income $\overline{R}_C$ (1), $K(t)$ is a certain function of t independent of ρ, k is the coefficient of homogeneity, and Θ represents the average

(1) § 228 and 233 above. For a constant value of ρ we have

$$K(t) = \alpha(t)\, e^{k'} \left[\frac{R_\omega(t)}{k}\right]^k$$

where R_ω represents the national primary income (§ 110), $\alpha(t)$ the effect of technical progress (§ 211), k the homogeneity coefficient (§ 211) and k' has the value indicated in § 228. For the exponential model we have [relation (251-17)]

$$K(t) = \alpha(t) \left[\frac{R_\omega(t)}{e\Theta_0}\right]^k$$

and for $k = 1$

$$K(t) = \frac{\alpha(t)\, R_\omega(t)}{e\Theta_0}$$

amortisation period for $i = \rho$. In other words, if we take the condition

(420-2) $$\rho \geqslant 0 .$$

Thus $\overline{R}_{CM}$ is maximised for $\rho = 0$.

This means that of all growth processes, the most advantageous is that for which the rate of increase of primary incomes is zero.

This apparently paradoxical result is easily explained when it is remembered that if population is increasing (and even supposing that natural resources increase at the same rate, *which they don't*), this growth implies that every year some proportion of gross national product must be allocated to maintain the volume per capita of equipment and durable consumption goods at the same level.

From the preceding relation, this diminution of consumable production is equal in relative terms to

(420-3) $$p = 1 - e^{-k \Theta_0 \rho}$$

and we have for small values of ρ

(420-4) $$p \sim k \Theta_0 \rho .$$

It further follows from relation (420-1) that a negative rate of growth (population decline) would be advantageous. This finding is valid so long as the assumptions, the consequence of which is the constancy of $(i - \rho)$, remain valid. But these assumptions would no longer be acceptable if the population declined to a point where sectors experiencing increasing returns became dominant (1).

The General Case

421. *These results are completely independent of the hypotheses concerning the function* $\beta(\theta)$ providing that the assumption be made that γ is constant, at least as a first approximation.

In fact, it has been shown that for any quasi-stationary process with steady growth, we have

$$\frac{1}{R}\frac{dR}{dt} = \frac{1}{C}\frac{dC}{dt} = \rho \qquad (421\text{-}1) \quad (2)$$

so that from relation (110-1) we have

$$R_C = (1 - \rho\,\gamma)R \qquad (421\text{-}2)$$

whence for $k = 1$ and at least as a first approximation (3)

$$\bar{R}_c = (1 - \rho\,\gamma)R \qquad (421\text{-}3)$$

(1) ALLAIS (1963): *Some Analytic and Applied Aspects of the Theory of Capital.*
(2) Relations (123-2) and (132-2).
(3) Relation (229-2).

and since from relations (251-12) and (251-6) we have

$$\gamma \sim \Theta_o \tag{421-4}$$

the preceding result relation (420-4) is found again.

Estimate of p *for the United States*

422. For the United States ρ and Θ_o may be taken as:

$$k = 1 \qquad \rho \sim 0.017 \qquad \Theta_o \sim 4 \qquad [1]$$

and according to (420-4)

$$p \sim 7\% \,.$$

The diminution of real consumed income is thus of the order of 6%.

(1) § 321 and 326.

C. — COMPARISON OF PRODUCTIVITY FOR TWO COUNTRIES

The Role of the Capitalistic Structure in the Relative Productivity of Two Countries

430. The theory presented enables the influence of the capitalistic structure of two countries on their relative productivity to be evaluated. For, from (410-1)

$$\frac{\overline{R}^1_c}{\overline{R}^2_c} = \frac{\overline{R}^1_{CM}}{\overline{R}^2_{CM}} \frac{\Theta_2}{\Theta_1} e^{\Theta_0\left(\frac{1}{\Theta_2} - \frac{1}{\Theta_1}\right)} \tag{430-1}$$

where $\overline{R}^1_c$ and $\overline{R}^2_c$ are the real national consumed incomes of the two countries considered.

According to (251-17) and for $k=1$ (§ 311) we have

$$\frac{\overline{R}^1_{CM}}{\overline{R}^2_{CM}} = \frac{R^1_\omega(t)}{R^2_\omega(t)} e^{-\Theta_0(\rho_1-\rho_2)} \tag{430-2}$$

where ρ^1 and ρ^2 represent the rates of growth of primary income R_ω for the two countries.

Finally we have

$$(430\text{-}3) \qquad \frac{\dfrac{\overline{R}^1_c}{R^1_\omega}}{\dfrac{\overline{R}^2_c}{R^2_\omega}} = e^{-\Theta_0(\rho_1-\rho_2)} \, \frac{\Theta_2}{\Theta_1} \, e^{\Theta_0\left[\frac{1}{\Theta_2}-\frac{1}{\Theta_1}\right]}$$

The first factor represents the relative influence of the growth of primary national income in the two countries (¹).

The second factor

$$(430\text{-}4) \qquad \frac{\Theta_2}{\Theta_1} \, e^{\Theta_0\left(\frac{1}{\Theta_2}-\frac{1}{\Theta_1}\right)}$$

represents the relative influence of the capitalistic structures of the two countries on the real consumed income per unit of primary income.

If Θ_1 and Θ_2 are of about the same values, it can be concluded that this influence is negligible.

This will be so if γ_1 and γ_2 have similar values, since

$$(430\text{-}5) \qquad \Theta = \frac{\Upsilon}{1-\rho\,\Upsilon} \qquad (^2)$$

and since ρ_1 and ρ_2 are generally small.

(¹) See § 420 to 422 above.

(²) Relation (251-11).

Comparative Productivity of the United States and French Economies

431. The comparison between the productivity of the French and American economies in 1957 can be summarized in the following figures:

	U.S.A. / FRANCE	
Productivity: ratio	2.3	
Equipment per worker: ratio . . .	2.4	
Output per unit of equipment: ratio	1	
	U.S.A.	France
Capital-Output Ratio $\gamma = C/R$	3.3 (1956)	> 3.3 (1) (1954)

From these figures it appears that output per working hour in the United States is approximately twice as high as in France, whereas the productivity of a unit of equipment is about the same (2).

Certain analyst have sought to attribute greater American productivity per man-hour to the larger volume of equipment used there. However, it has been seen that

$$\gamma = \frac{C}{R} = \hat{\Theta} \qquad (431\text{-}1) \qquad (3)$$

(1) As far as can be judged, γ is of the same order of magnitude in France as in the U.S.A. [ALLAIS (1960 B), p. 29].
(2) ALLAIS (1960 B), pp. 28-32 and 195-297; ALLAIS (1962 A), p. 721.
(3) Relation (251-12).

Since the French and American economies are characterized by values of γ of the same order of magnitude, it may be concluded that the capitalistic structures of the two countries are similar (1); they can therefore not play any role in accounting for the average difference in productivity observed.

This average difference of productivity, of the order of 2.3 to 1, should be attributed to other causes (2). It follows then that with the same capital-output ratio $\gamma = C/R$, the amount of equipment per worker is approximately double.

In conclusion, the fact that the ratio between the physical volume of industrial equipment per worker is 2.4 to 1 whereas the value of capital relative to the national income is little different, should not be considered as a cause in itself, but as an effect of greater American efficiency. The fact that *in physical quantity* American equipment per head is at present 2.4 times as high as in France corresponds simply to the fact that, *for reasons other than the capitalistic structure,* U.S.A. productivity is about 2.3 times as high. It follows from this that for the same value of capital per head measured in terms of hours of work, the amount of equipment is about 2.3 times as high. From this point of view, the concordance of the order of magnitude of the two values 2.3 and 2.4, which were found as the ratios of productivity and of the volume of output respectively, is particularly striking.

Thus, if an explanation of greater American efficiency is sought using the greater value of equipment per worker measured in physical units rather then by considering the capital-output ratio as a means of estimating the influence of accumulated capital, *this is to treat as a cause a phenomenon which in reality is only an effect* (3).

(1) Since they are characterised by the same function $\varphi(0)$ and $\hat{\varphi}(0)$, at least as a first approximation (§ III).

(2) ALLAIS (1960 B), Part I.

(3) ALLAIS (1948) and (1960 B), pp. 28-32.

Analysis of a Simple Model

432. These considerations can be illustrated by a simple model which shows that the conventional explanation of greater U.S. productivity as being due to the larger volume of equipment per worker *may* be completely erroneous.

Consider an economy in stationary equilibrium characterised by the three following relationships:

(432-1)
$$\begin{cases} A = f\left[qX_A, E\right] \\ E = g\left[qX_E\right] \\ X_A + X_E = X \end{cases}$$

A is the quantity of consumption good (A) produced during each period, E the output of an equipment good (E), X total labour input of which a part X_A is absorbed by industry (A) while X_E represents the labour input of industry E.

The quantity E of equipment goods produced during a preceding period is assumed to be used completely during the next period.

Finally, q is *a parameter indicating quality.*

The functions f and g are increasing functions of each of the variables and are assumed to be of *first order* homogeneity.

We now consider two situations I and II characterised respectively by two values q_1 and q_2 of q with

(432-2)
$$q_2 = 2\, q_1$$

the values of X_A and X_E remaining unchanged.

Since the functions are assumed to be of first-order homogeneity, we have

(432-3) $$\frac{A_2}{A_1} = \frac{E_2}{E_1} = \frac{q_2}{q_1} = 2$$

In situation II

(432-4) $$A_2 = 2\ A_1$$

because of two effects, one direct, since

(432-5) $$q_2\ X_A = 2\ q_1\ X_E$$

the other indirect, due to

(432-6) $$E_2 = 2\ E_1\ .$$

The following table summarises the comparison between the two situations

	Situation II / Situation I
Productivity: ratio	$A_2/A_1 = q_2/q_1 = 2$
Equipment per worker: ratio . . .	$E_2/E_1 = q_2/q_1 = 2$
Output per unit of equipment: ratio	$A_2/E_2/A_1/E_1 = 1$
Capital-Output Ratio $\gamma = C/R$	$\gamma_2 = \gamma_1$

The case described here is completely analogous to the American/French case examined above, and the fact that the similarity is formalistic does not render it any less striking.

If he had no other facts than these, an economist from another planet with no knowledge of the internal structure of the model ([1]) would be tempted to aver that the higher productivity in situation II resulted from the larger volume of equipment in use.

But, *by hypothesis,* we know that such an explanation woud be *totally* misleading, since the higher productivity in situation II is entirely due to the greater value $q_2 = 2q_1$ of the parameter q.

If, in this model, productivity is twice as high in situation II, this is not because there is twice as much equipment available, but because of a factor other than the capitalistic structure. The fact that the volume of equipment per worker is twice as high follows from the fact that productivity is twice as high while the capital-output ratio is the same.

In reality, it can be verified from the model that the volume of equipment plays *an intermediate* role, and to clarify the issue it should be left out of account. From relations (432-1) we then get

$$(432\text{-}7) \qquad A = f[qX_A,\ g(qX_E)]$$

whence, because of the first-order homogeneity

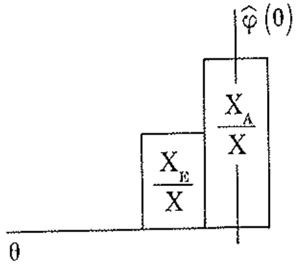
$$(432\text{-}8) \qquad \frac{A}{X} = qF\left[\frac{X_A}{X}\ ,\ \frac{X_E}{X}\right]$$

([1]) Characterised in particular by equation (432-2).

Thus we see that *in the model considered,* productivity depends only on q and on the coefficients X_A/X and X_E/E which are moreover related to each other by

(432-9) $$\frac{X_A}{X} + \frac{X_E}{X} = 1$$

As far as capital is concerned, productivity in the model is proportional to q and from the capitalistic point of view depends only on the allocation of labour between direct (X_A) and indirect (X_E) activity. This allocation is represented by the « characteristic diagram »



This diagram, and not the volume of equipment, characterises the capitalistic structure (1).

As between the two situations I and II above, the capitalistic structure is the same.

It would, of course, be inaccurate to claim that the volume of equipment is irrelevant to the higher productivity of situation II, since it enters as an essential intermediary; but it would be no less inaccurate to claim that the twofold productivity was due to the use of a doubled volume of equipment.

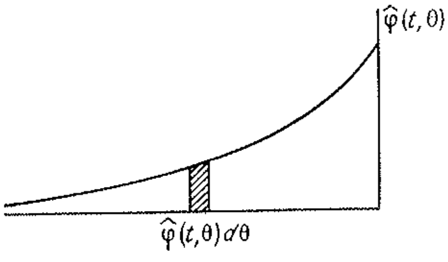
(1) § III.

Naturally, this model proves *nothing* so far as the analysis of the comparative productivity of the American and French economies is concerned [1]. It shows merely that a type of reasoning which is generally considered as very attractive *may* be completely false.

Analysis of the Difference in Productivity between the French and American Economies

433. *Equipment,* as we have just seen, *plays only an intermediate role* in the analysis of productivity differences between the American and French economies. *It is the capitalistic structure which it is important to study.*

The capitalistic structure is represented by its characteristic function



which describes the time distribution of the inputs of primary income used in the past emerging in the consumed national income of instant t. There is every reason to believe that this

[1] I should particularly stress that in my opinion greater American productivity *cannot be explained by a higher quality of the labour force there.* (See ALLAIS, 1960 B, p. 33).

curve is of exponential form [1]. If so, it can be characterised by a single coefficient, the average distance in time of the primary inputs

$$(433\text{-}1) \qquad \hat{\Theta} = \int_0^{\infty} \theta \, \hat{\varphi}(\theta) \, d\theta \qquad [2]$$

whence it follows that if the values of $\hat{\Theta} = \gamma$ [3] are approximately the same for the two countries, their capitalistic structures are roughly similar and therefore cannot represent an explanatory factor of greater American productivity.

(1) Relation 251-4) and § 336.
(2) Relation (112-2).
(3) Relation (251-12).

D. — POLICIES FOR CAPITALISTIC DEVELOPMENT

440. Productivity growth is an essential component of any development programme. This growth may result from various factors, such as an improvement of technical knowledge, better training of executives and workers, etc. *The purpose of the present analysis is to estimate the influence on productivity of the more or less capitalistic economic structure, in other words, variations in the extent to which more or less use is made of indirect production processes.*

Evaluated in terms of years of national income, the capital-output ratio in Africa or Asia is probably not of a very different order of magnitude fro mits value in the United States. Even assuming a value of only 2 (which seems very unlikely) (1) against 3.5 for the U.S.A., this could not explain more than a productivity difference of the order of 1 to 2, whereas the difference is at least of 1 to 20. Thus, if American production techniques were abruptly trausplanted in Africa (which would require enormous investment) without at the same time remedying the existing causes of low productivity, the African economy would not be able to replace completely the investments which had been made in parallel with their depreciation and obsolescence.

In fact, it has already been noted that the model implies the apparently paradoxical relation

$$C = \Theta_0 \, R_\omega \qquad (2)$$

(1) Allais, 1960 A*, § 25.
(2) Relation (251-9).

which nevertheless on the basis of the information available at the present time appears to be verified by the available statistical data (1).

As we have seen the economic significance of this relation is that the value in wage units of reproducible capital does not depend on the more or less capitalistic (i.e. more or less roundabout) structure of the production process, as described by the coefficient Θ. The value in wage units of available capital is independent of the rate of interest i and of the rate of growth ρ.

This means that, evaluated in hours of work, the figures for capital available per worker in so-called under-developed economies should be comparable to the figures of the advanced economies (1).

The implication of this is that the accumulated capital of the advanced economies does not have any greater weight for them than that represented for under-developed economies by the capital they have available.

This finding shows that in formulating development policy, it may be advantageous to adopt the most efficient production techniques as quickly as possible. In this case, an attempt should be made to keep interest rates so low that more roundabout techniques can be applied.

However, it follows clearly from the preceding that the more or less capitalistic structure of the production process (Θ smaller or greater) does not suffice to explain recorded differences in productivity levels.

In reality, the explanation of the enormous differences in productivity observed as between the west and underdeveloped countries ,half of the world's population has an income less than 1/20th that the average American (2) has much less to do with below standard values of the capital-output ratio than with:

a) differences in per capita availability of natural resources,

(1) See Allais (1960 A), § 25.
(2) Allais (1961 B) vol. I (n. 8), pp. 22-24 and vol. II (n. 7), pp. 39-80.

b) differences in the level of technical education,

c) differences in economic management in general.

If follows immediately that the installation in Africa and Asia of equipment analogous to that of the U.S.A. would not be sufficient unless ways were found to deal with the other factors which make for these countries' lower productivity (1).

And this leads in turn to the conclusion that the general emphasis placed in recent years on savings and investment as the key to speedy development, a view held in respect of Europe as much as for third countries, is based on an erroneous position. *The factors essential to development are completely different.*

Fortunately, it appears that this point of view, which the author has propounded continuously since 1947, is now shoved more or less explicity, by an ever growing number of economists. The present study basically constitutes its theoretical justification.

(1) ALLAIS (1961 B) and (1962 A).

APPENDIX

THE INFUENCE ON THE RESULTS OF A VARIATION OF THE FUNCTION $\beta(\theta)$

It is very interesting to study what happens to the various results when the function $\beta(\theta)$ is no longer of exponential form, the aim being to see to what extent they are modified when that function changes.

I will study first the general case in which, μ being constant, $\beta(\theta)e^{\mu\theta}$ can be expanded as a Taylor series and second two particular cases for which the calculations can be carried all the way through.

It may perhaps be thought that the calculations of this appendix are no more than an intellectual game. *Yet they are very important in the sense that they allow the demonstration of the fact that under very general hypotheses, the parameter Δ (1) does not differ very greatly from unity, whence it follows that the properties of the general model remain very similar to those of the exponential model.*

(1) Relations (220-6), (220-14), (251-17) and § 314 and 335.

I. — THE GENERAL CASE
$\beta(\theta)e^{\mu\theta}$ CAN BE DEVELOPED AS A TAYLOR SERIES

Hypothesis: The Development of $\beta(\theta)e^{\mu\theta}$ as a Taylor Series

510. Assume that the function $\beta(\theta)e^{\mu\theta}$, in which μ is a certain constant, can be developed as a Taylor series ([1]). This is a very general hypothesis. Then

$$\beta(\theta)=\left[b_0+b_1\theta+\dots+\frac{b_{n-1}\theta^{n-1}}{\overline{n-1}!}+\dots\right]e^{-\mu\theta} \tag{510-1}$$

Since

$$k=\int_0^\infty \beta(\theta)\,d\theta \qquad \psi(u)=\frac{1}{k}\int_0^\infty \beta(\theta)e^{-u\theta}\,d\theta \qquad \text{([2])} \tag{510-2}$$

we have

$$k=\frac{b_0}{\mu}+\frac{b_1}{\mu^2}+\dots+\frac{b_{n-1}}{\mu^n}+\dots \tag{510-3}$$

$$\psi(u)=\frac{1}{k}\left[\frac{b_0}{\mu+u}+\frac{b_1}{(\mu+u)^2}+\dots+\frac{b_{n-1}}{(\mu+u)^n}+\dots\right] \tag{510-4}$$

([1]) It goes without saying that for *practical purposes* it is only of use to consider an interval for θ of 0 to 25 years, or at most of 0 to 100 years.
([2]) Relations (211-3) and (220-1).

This expression for ψ enables all the relations found in the case of the general model to be applied.

We have in particular [1]

$$(510\text{-}5) \qquad \varphi(\theta) = \left[b_0 + b_1 \theta + \dots + \frac{b_{n-1}\,\theta^{n-1}}{\overline{n-1}!} + \dots \right] \frac{e^{-(\mu+i-\rho)\theta}}{k\,\psi(i-\rho)}$$

The Meaning of the b_{n-1}

The hypothesis made is equivalent to assuming that

$$(510\text{-}6) \qquad \beta(\theta) = \omega_0\,\beta_0(\theta) + \omega_1\,\beta_1(\theta) + \dots + \omega_{n-1}\,\beta_{n-1}(\theta) + \dots$$

with

$$(510\text{-}7) \qquad \beta_{n-1} = \frac{k}{\overline{n-1}!}\left(\frac{\theta}{T}\right)^{n-1} \frac{e^{-\frac{\theta}{T}}}{T}$$

$$(510\text{-}8) \qquad \mu = \frac{1}{T}$$

$$(510\text{-}9) \qquad \omega_{n-1} = \frac{1}{k}\,\frac{b_{n-1}}{\mu^n} = \frac{b_{n-1}}{k}\,T^n$$

(1) Relation (223-7).

$$\omega_0 + \omega_1 + \cdots + \omega_{n-1} + \cdots = 1 \tag{510-10}$$

$$\int_0^\infty \beta_{n-1}(\theta)\, d\,\theta = k \tag{510-11}$$

The function $\beta(\theta)$ *then appears as the weighted average of the functions* $\beta_{n-1}(\theta)$, *each satisfying the condition that its integral over the range zero to infinity is equal to* k, *and that the sum of the weights is unity.*

Naturally we have

$$\beta_0 = k\, \frac{e^{-\frac{\theta}{T}}}{T} \tag{510-12}$$

Thus, the larger ω_0, *the more closely* $\beta(\theta)$ *approximates to an exponential form.*

Convergence of the Series Expansions

It can be seen directly from the expression for the functions $\beta_{n-1}(\theta)$ that the convergence of their series expansions depends *essentially* on the order of magnitude of T. The smaller T, the more rapidly the exponentials converge.

From this point of view, the order of magnitude of the ω_{n-1} is of secondary importance.

The Sign of the ω_{n-1}

The ω_{n-1} may be positive or negative. However, to simplify the exposition, the following discussion will be limited to the case in which they are positive. It will be seen, however, that the majority of the results which will be derived, in particular those concerning the order of magnitude of the parameters T, Θ_0 and Δ, hold in the general case where some ω_{n-1} coefficient have a negative value.

Expression for $\psi(u)$ *and* $\varphi(\theta)$

511. In the same way we have from (510-2) and (510-6)

(511-1) $$\psi(u) = \omega_0 \psi_0(u) + \omega_1 \psi_1(u) + \dots + \omega_{n-1} \psi_{n-1}(u) + \dots$$

with

(511-2) $$\psi_{n-1}(u) = \frac{\mu^n}{(\mu + u)^n} = \frac{1}{(1+\mathrm{T}u)^n}$$

Finally, from the general formula (223-7) we have

(511-3) $$\varphi(\theta) = \frac{\beta(\theta)\, e^{-(i-\rho)\theta}}{k\,\psi(i-\rho)}$$

Thus if we put

$$(5\text{II-}4) \qquad \varphi_{n-1}(\theta) = \frac{\beta_{n-1}(\theta)\, e^{-(i-\rho)\theta}}{k\,\psi_{n-1}(i-\rho)}$$

then

$$(5\text{II-}5) \quad \varphi(\theta) = \frac{\omega_0\,\psi_0(i-\rho)\,\varphi_0(\theta) + \ldots + \omega_{n-1}\,\psi_{n-1}(i-\rho)\,\varphi_{n-1}(\theta) + \ldots}{\psi(i-\rho)}$$

i.e.

$$(5\text{II-}6) \quad \varphi(\theta) = \frac{\omega_0\,\psi_0(i-\rho)\,\varphi_0(\theta) + \ldots + \omega_{n-1}\,\psi_{n-1}(i-\rho)\,\varphi_{n-1}(\theta) + \ldots}{\omega_0\,\psi_0(i-\rho) + \ldots + \omega_{n-1}\,\psi_{n-1}(i-\rho) + \ldots}$$

The function $\varphi(\theta)$ thus appears as a weighted average of the function $\varphi_{n-1}(\theta)$ with weights $\omega_{n-1}\,\varphi_{n-1}$.

We have

$$(5\text{II-}7) \qquad \psi_{n-1}(i-\rho) = \frac{\text{I}}{\left(\text{I} + \frac{i-\rho}{\mu}\right)^n}$$

Thus providing $(i-\rho)$ is small and n not too large, the weighting of the functions $\varphi_{n-1}(\theta)$ will differ little from the weighting of the functions $\psi_{n-1}(\theta)$.

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Representation of an Amortization Law $\varphi(\theta)$ by a Taylor Series Expansion

It is clear that *for a given interval of variation of* θ, *for example*

$$0 < \theta < 100 \text{ years}$$

it is always possible to represent any function $\varphi(\theta)$ *given by the statistical analysis with a sufficient degree of approximation by taking a limited number* p *of the terms of the Taylor expansion of* $\beta(\theta)e^{\mu\theta}$. *Thus it is essential to study the properties of this expansion.*

Expressions for Θ_0 *and* Δ

512. Developing the different terms of $\varphi(u)$ in series (1), we have

(512-1)
$$\frac{b_{n-1}}{(\mu+u)^n} = \frac{b_{n-1}}{\mu^n} \frac{1}{\left(1+\frac{u}{\mu}\right)^n}$$

$$= \frac{b_{n-1}}{\mu^n}\left[1-n\frac{u}{\mu}+\frac{n(n-1)}{2}\frac{u^2}{\mu^2}+\cdots\right]$$

(1) This is legitimate if $u/\mu<1$, a condition which is usually met by observed values of $i-\rho$. According to (510-8) we have $\mu=1/T$ and T is given below by (512-8). We will see that in any case we have $T<\Theta_0$ [relation (513-2)].

whence

$$(512\text{-}2) \qquad k\,\psi(u) = \frac{b_0}{\mu}\left(1 - \frac{u}{\mu} + \frac{u^2}{\mu^2}\cdots\right)$$

$$+ \frac{b_1}{\mu^2}\left(1 - \frac{2u}{\mu} + \frac{3u^2}{\mu^2}\cdots\right)$$

$$\cdots\cdots\cdots\cdots\cdots\cdots$$

$$+ \frac{b_{n-1}}{\mu^n}\left(1 - n\frac{u}{\mu} + \frac{n(n+1)}{2}\frac{u^2}{\mu^2} + \cdots\right)$$

$$\cdots\cdots\cdots\cdots\cdots\cdots$$

Since

$$(512\text{-}3) \qquad \psi(u) = 1 - \Theta_0\, u + \Delta\,\Theta_0^2\, u^2 \ldots\ldots \qquad (^1)$$

we have

$$(512\text{-}4) \qquad \Theta_0 = \frac{1}{k}\,\frac{1}{\mu}\left[\frac{b_0}{\mu} + \frac{2b_1}{\mu^2} + \cdots\cdots + \frac{n\,b_{n-1}}{\mu^n} + \cdots\right]$$

$$(512\text{-}5) \qquad \Delta = k\,\frac{\left[\dfrac{b_0}{\mu} + \dfrac{3b_1}{\mu^2} + \cdots\cdots + \dfrac{n(n-1)}{2}\,\dfrac{b_{n-1}}{\mu^n} + \cdots\right]}{\left[\dfrac{b_0}{\mu} + \dfrac{2b_1}{\mu^2} + \cdots\cdots + n\,\dfrac{b_{n-1}}{\mu^n} + \cdots\right]^2}$$

(1) Main text, relation (220-2).

so that

(512-6) $$\frac{\Theta_0}{T} = \omega_0 + 2\,\omega_1 + \cdots + n\,\omega_{n-1} + \cdots$$

(512-7) $$\Delta = \frac{\omega_0 + 3\,\omega_1 + \cdots + \frac{n(n+1)}{2}\,\omega_{n-1} + \cdots}{\left(\omega_0 + 2\,\omega_1 + \cdots + n\,\omega_{n-1} + \cdots\right)^2}$$

or again

(512-8) $$\frac{\Theta_0}{T} = \sum_{n=1}^{\infty} n\,\omega_{n-1}$$

(512-9) $$\Delta = \frac{\sum_{n=1}^{n=\infty} \omega_{n-1} \frac{n(n+1)}{2}}{\left[\sum_{n=1}^{n=\infty} \omega_{n-1}\, n\right]^2}$$

subject, as I have shown, to the condition

(512-10) $$\sum_{n=1}^{\infty} \omega_{n-1} = 1$$

Further, taking the values of the global quantities R, R_c, C, γ_c, etc., as they are given by the statistical analysis is equivalent to fixing the value of $\psi(i-\rho)$ for a given value of

$i-\rho$ (1). Thus, from (511-1) and (511-2) the values of T and of the ω must satisfy the condition

$$(512\text{-}11) \quad \frac{\omega_0}{1+T(i-\rho)}+\frac{\omega_1}{[1+T(i-\rho)]^2}+\cdots+\frac{\omega_{n-1}}{[1+T(i-\rho)]^n}+\cdots=\psi(i-\rho)$$

where the value of $\psi(i-\rho)$ is given.

Thus, the weights ω_{n-1} are subject to the two conditions (510-10) and (512-11). For given weights ω_{n-1} satisfying the condition (512-10), the relation (512-11) determines T and the relation (512-9) determines Δ. Relation (512-8) determines Θ_0 once T is known.

As the expression for Δ as a function of the ω_{n-1} is independent of T, discussion of the value of Δ can be carried on independently of the conditions (512-8) and (512-11), taking account only of the condition (512-9).

Equation (512-9) can be rewritten as follows (for simplicity the subscripts of the Σ have been omitted)

$$(512\text{-}12) \qquad \Delta=\frac{\Sigma n^2\,\omega_{n-1}\,\Delta_{n-1}}{\left[\Sigma n\ \omega_{n-1}\right]^2} \qquad \text{for} \quad n\geqslant 1$$

with

$$(512\text{-}13) \qquad \frac{1}{2}<\Delta_{n-1}=\frac{1}{2}\left(1+\frac{1}{n}\right)\leqslant 1$$

where Δ_{n-1} denotes the value of Δ for the function β_{n-1} (2).

(1) § 312.
For example, we have

$$\gamma_c=\frac{1-\psi(i-\rho)}{i-\rho}$$

(relation 226-2).

(2) Relations (220-6) and (510-7).

Again, we have

(512-14) $$\Delta = \frac{1}{2} \frac{1}{\Sigma\, \omega_{n-1} n} + \frac{1}{2} \frac{\Sigma\, \omega_{n-1} n^2}{\left[\Sigma\, \omega_{n-1}\, n\right]^2}$$

As the first term is positive and the second is necessarily greater than 1/2 (1), we again find the general condition

(512-15) $$\Delta \geqslant 1/2$$ (2)

The Order of Magnitude of T

513. It may be remarked that in general, and if the difference $(i - \rho)$ is sufficiently small, if γ_C is given (3), this is equivalent, *as a first approximation,* to the Θ_o being given (4). It follows that the order of magnitude of T is determined, as a first approximation, by the relation

(513-1) $$T \sim \frac{\gamma_C}{\sum_{n-1}^{\infty} n\, \omega_{n-1}}$$ (5)

Consequently the less rapid the decline of the coefficients ω_{n-1}, the lower the value of T, and the more rapid the convergence of the series expansions (512-1).

(1) From the BUNJAKOWSKY-SCHWARZ Inequality (BRONSTEIN and SEMENDJAJEW, *Taschenbuch der Mathematik*, p. 135).

This inequality, moreover, follows immediately from the interpretation given in § 517 below (relation 517-2).

(2) Main text, condition (220-13).

(3) Which, it may again be recalled, is equivalent to $\phi(i-\rho)$ being given (relation 226-2).

(4) We have according to relation (240-10).

(2) $$\gamma_C \backsim \Theta_o\, [1 - \Delta\, \Theta_o\, (i-\rho)]\ .$$

(5) Relation (512-8).

In any case, we have according to (512-8) and (512-10)

(513-2) $$T < \Theta_o .$$

Order of Magnitude of Δ

1) *Case in which the* ω_{n-1} *decline sufficiently quickly*

514. In the first place, it can easily be seen that *if the* ω_{n-1} *decline sufficiently rapidly,* Δ *will have a value not very different from unity.*

a) *Decrease of the* ω_{n-1} *as* $1/n^q$ $(q \geqslant 4)$

Assume first that the ω_{n-1} decline as with $1/n^q$ so that for example

(514-1) $$\omega_{n-1} = \frac{\lambda}{n^q}$$

From (512-14) we will have

(514-2) $$\Delta = \Delta q = \frac{1}{2\lambda}\left[\frac{1}{\sum \frac{1}{n^{q-1}}} + \frac{\frac{1}{\sum n^{q-2}}}{\left(\sum \frac{1}{n^{q-1}}\right)^2}\right]$$

with

(514-3) $$\lambda \sum \frac{1}{n^q} = 1$$

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We will then have

(514-4) $$\Delta_q = \frac{\zeta(q)}{2}\left[\frac{1}{\zeta(q-1)} + \frac{\zeta(q-2)}{[\zeta(q-1)]^2}\right]$$

where $\zeta(p)$ is the Riemann function

(514-5) $$\zeta(p) = 1 + \frac{1}{2^p} + \frac{1}{3^p} + \cdots + \frac{1}{n^p} + \cdots$$

As $\zeta(p)$ is finite for $p \geqslant 2$, Δq is finite for $q \geqslant 4$.
For $q = 4$, we have

$$\Delta_4 \sim 1.06$$

Δq is a decreasing function of q and we have

$$\Delta_\infty = 1$$

so that for the particular hypothesis envisaged,

(514-6) $$1 < \Delta < 1.06$$

These results are of course valid when the $\lambda_{n-1} = n^q \omega_{n-1}$ corresponding to equation (514-1), although unequal, are nevertheless of the same order of magnitude.

b) *Exponential Decrease of the* ω_{n-1}

515. Now assume instead that ω_{n-1} diminishes as $e^{-K(n-1)}$, so that for example

(515-1) $$\omega_{n-1} = \lambda e^{-K(n-1)}$$

Then we will have

$$
(515\text{-}2)\quad
\left\{
\begin{aligned}
1=\Sigma\omega_{n-1} &= \lambda\left[1+e^{-K}+e^{-2K}+\ldots\right] \\
&= \frac{\lambda}{1-e^{K}} \\
\Sigma n\,\omega_{n-1} &= \lambda\left[1+2e^{-K}+3e^{-2K}+\ldots\right] \\
&= \frac{\lambda}{e^{-K}}\left[e^{-K}+2e^{-2K}+3e^{-3K}+\ldots\right] \\
&= -\frac{\lambda}{e^{-K}}\,\frac{d}{dK}\,\frac{1}{1-e^{-K}} \\
&= \frac{1}{\left(1-e^{-K}\right)} \\
\Sigma n^{2}\omega_{n-1} &= \lambda\left[1+4e^{-K}+9e^{-2K}+\ldots\right] \\
&= \frac{\lambda}{e^{-K}}\left[e^{-K}+4e^{-2K}+9e^{-3K}+\ldots\right] \\
&= \frac{\lambda}{e^{-K}}\,\frac{d^{2}}{dK^{2}}\,\frac{1}{1-e^{-K}} \\
&= \frac{1+e^{-K}}{\left[1-e^{-K}\right]^{2}}
\end{aligned}
\right.
$$

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so that according to (512-14)

$$(515\text{-}3) \qquad \Delta = \frac{1}{2}\left[1 - e^{-K} + \frac{1+e^{-K}}{\left(1-e^{-K}\right)^2}\left(1-e^{-K}\right)^2\right]$$

i.e.

$$(515\text{-}4) \qquad \Delta = 1$$

whatever the value of K. This result is a very remarkable one; *it is due to the fact that in this case the model reduces to the exponential model.*

For, from (511-1) we have

$$(515\text{-}5) \qquad \begin{aligned} \varphi(u) &= \left(1-e^{-K}\right)\left[\frac{1}{1+Tu} + \cdots\cdots + \frac{e^{-K(n-1)}}{(1+Tu)^n} + \cdots\right] \\ &= \frac{1-e^{-K}}{(1+Tu)\left(1-\dfrac{e^{-K}}{1+Tu}\right)} \end{aligned}$$

which gives finally

$$(515\text{-}6) \qquad \varphi(u) = \frac{1}{1+\dfrac{Tu}{1-e^{-K}}}$$

which is effectively the function ψ of the exponential model with

$$\Theta_0 = \frac{T}{1-e^{-K}} \tag{515-7}$$ (1)

Furthermore, from (510-6) we have

$$\beta(\theta) = k(1-e^{-K})\left[1+e^{-K}\frac{\theta}{T}+\dots+\frac{e^{-K(n-1)}}{\overline{n-1}!}\left(\frac{\theta}{T}\right)^{n-1}+\dots\right]\frac{e^{-\frac{\theta}{T}}}{T} \tag{515-8}$$

i.e.

$$\beta(\theta) = k(1-e^{-K})e^{\frac{e^{-K}\theta}{T}}\frac{e^{-\frac{\theta}{T}}}{T} \tag{515-9}$$ (2)

or finally, taking (515-7) into account

$$\beta(\theta) = k\frac{e^{-\frac{\theta}{\Theta_0}}}{\Theta_0} \tag{515-10}$$

which is effectively the function $\beta(\theta)$ of the exponential model (3).

Naturally, these results will hold approximately if the λ_{n-1} of the equation (515-1), although not equal, are nevertheless of the same order of magnitude.

(1) Relation (251-1).

(2) Since

$$e^X = 1 + X + \dots + \frac{X^{n-1}}{\overline{n-1}\,!} + \dots$$

(3) Relation (351-2).

c) *Decline of the* ω_{n-1} *as* $n^q e^{-K(n-1)}$

Again, it could be assumed that

(515-11) $$\omega_{n-1} = \lambda n^q e^{-K(n-1)}$$

For $q = 1$ and using the same methods as above, we find for example

(515-12) $$1 = \Sigma\, \omega_{n-1} = \frac{\lambda}{(1-e^{-K})^2}$$

(515-13) $$\Sigma n\, \omega_{n-1} = \lambda\left[1 + 4e^{-K} + 9e^{-2K} + \dots\right]$$

$$= \frac{1+e^{-K}}{1-e^{-K}}$$

(515-14) $$\Sigma n^2\, \omega_{n-1} = \lambda\left[1 + 8e^{-K} + 27e^{-2K} + \dots\right]$$

$$= \frac{\lambda}{e^K}\,\frac{d^3}{dK^3}\,\frac{1}{(1-e^{-K})}$$

$$= \frac{1+4e^{-K}+e^{-2K}}{(1-e^{-K})^2}$$

so that, according to (512-14)

(515-15) $$\Delta = \frac{1+2e^{-K}}{(1+e^{-K})^2}$$

When K grows from zero to infinity, Δ grows from 3/4 to 1, so that

$$(515\text{-}16) \qquad 3/4 < \Delta < 1 \,.$$

In all cases it can be seen that a sufficiently rapid decline in the ω_{n-1} will result in a value of Δ relatively near to 1.

From this, it may be deduced that in this case the properties of the model will not differ greatly from those of the exponential model.

Case in which the ω_{n-1} vanish beyond a Certain Rank

516. Now examine the case in which the ω_{n-1} vanish beyond a certain rank and assume that

$$(516\text{-}1) \qquad \omega_{n-1} = 0 \qquad \text{for} \qquad n \geqslant p+1 \,.$$

a) *Case in which all ω_{n-1} are of the same order of magnitude*

Assume firstly that all ω_{n-1} are of the same order of magnitude, for example, that they are all equal.

Then from relation (512-14)

$$(516\text{-}2) \qquad \Delta = \frac{1}{2}\left[\frac{1}{\frac{1}{p}\,\frac{p(p+1)}{2}} + \frac{\frac{1}{p}\,\frac{p(p+1)(2p+1)}{6}}{\left(\frac{p+1}{2}\right)^2}\right]$$

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i.e.

(516-3) $$\Delta = \frac{2}{3}\left(1 + \frac{1}{p+1}\right)$$

we have thus

(516-4) $$\frac{2}{3} < \Delta \leqslant 1$$

Δ being equal to 1 for $p = 1$ (exponential model).

An analogous result holds of course if the ω_{n-1}, although not equal, are nevertheless of the same order of magnitude.

b) *Case in which the* ω_{n-1} *do not decline rapidly*

517. It is again easy to see that *if the quantities* ω_{n-1} *do not decrease too rapidly,* Δ *is below unity or very near to it.*

For, in this hypothesis and from (512-14), $(1/[2\Sigma n\omega_{n-1}])$ which is in any case below $1/2$, will generally be small.

As to the second term of (512-14), if the ω_{n-1}, are considered as masses and the n as distances, it can be written

(517-1) $$\frac{1}{2}\,\frac{\Sigma m_n\,\mathrm{OM}_n^2}{[\Sigma m_n \mathrm{OM}_n]^2}$$

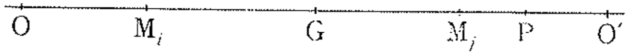
so that

(517-2) $$\frac{1}{2}\,\frac{\mathrm{OG}^2\Sigma m_n + \Sigma m_n\,\mathrm{GM}_n^2}{\mathrm{OG}^2\,\Sigma m_n}$$

$$= \frac{1}{2} + \frac{1}{2}\,\frac{\Sigma m_n\,\mathrm{GM}_n^2}{\mathrm{OG}^2\Sigma m_n}$$

in which G is the centre of gravity of the points M_n, assumed to be of mass m_n.

But the points M_n can be classified in two categories M_i and M_j according to whether they lie between O and G or between G and P of the abcissa p



For the M_i points

(517-3) $$GM_i < OG \, .$$

Again, if the ω_{n-1} *do not decline too rapidly,* G will be to the right of the centre of the segment OP, and the image O′ of O with respect to G will be to the right of P. Then we will have

(517-4) $$G\, M_j < O\, G$$

so that

(517-5) $$\frac{1}{2} \, \frac{\Sigma m_n \, GM_n^2}{OG^2 \Sigma m_n} < \frac{1}{2}$$

Thus under the assumptions made, that is to say if

(517-6) $$\frac{\sum_{n=1}^{n=p} \omega_{n-1} \, n}{\sum_{n=1}^{n=p} \omega_{n-1}} > \frac{p}{2}$$

[11] *Allais* - pág. 224

we have certainly from (512-14)

$$(517\text{-}7) \qquad \frac{1}{2} < \Delta < 1 + \frac{1}{p}$$

and since $\delta = \Delta - 1$ [1]

$$(517\text{-}8) \qquad -\frac{1}{2} < \delta < \frac{1}{p} \qquad [2]$$

In an apparently paradoxical fashion we see that for this condition to hold, the coefficients of high powers of θ in the expansion of $\beta((\theta)$ and $\beta(\theta)e^{\mu\theta}$ must be sufficiently large.

Naturally, it follows from the foregoing *that in the general case, in which the* ω_{n-1} *vanish only at infinity, the conditions* (517-8) *will again be verified if*

$$(517\text{-}9) \qquad \frac{\sum_{n=1}^{n=\infty} \omega_{n-1}\, GM_n^2}{OG^2 \sum_{n=1}^{n=\infty} \omega_{n-1}} < 1$$

which is a relatively weak condition.

Finally, we see that in the case of hypothesis (517-6), *as in the case of the weaker hypothesis* (517-9), *the inequalities*

---

(1) Relation (220-14).

(2) In the light of the indications given above δ will probably be negative.

(517-8) *are verified, whence it follows that the limited expansions derived in the case of the general model for the different quantities, and given in* § 240, *may be used.*

518. Secondly, since $n \geqslant 1$, it follows from (512-12), (512-13) and (512-14) that

$$\frac{1}{2} \frac{\Sigma n^2 \omega_{n-1}}{\left[\Sigma n \; \omega_{n-1}\right]^2} < \Delta < \frac{\Sigma n^2 \omega_{n-1}}{\left[\Sigma n \; \omega_{n-1}\right]^2} \tag{518-1}$$

Assume that

$$\omega_{n-1} = 0 \qquad \text{for} \qquad n - 1 \geqslant p \tag{518-2}$$

so that the sums only cover p terms. Assuming further that the $n\omega_{n-1}$ are never-increasing, then applying the Tchebicheff inequality (1)

$$\Sigma n^2 \omega_{n-1} \leqslant \frac{1}{p} \sum_{n=1}^{n=p} n \sum_{n=1}^{n=p} n \, \omega_{n-1} \tag{518-3}$$

whence

$$\Delta < \frac{1}{p} \frac{\Sigma n}{\Sigma n \; \omega_{n-1}} \tag{518-4}$$

i.e.

$$\Delta < \frac{1}{2} \frac{p+1}{\sum_{n=1}^{n=p} n \, \omega_{n-1}} \tag{518-5}$$

(1) BRONSTEIN-SEMENDJAJEW, *op. cit.*, p. 135.

This condition shows that if the ω_{n-1} do not decline too rapidly, the upper limit of Δ will remain relatively low.

If, for example, we assume that

(518-6) $$n\omega_{n-1} = K(1 - \omega_0) \quad \text{for} \quad n \geqslant 2$$

then we will have from (512-10)

(518-7) $$\Sigma\, \omega_{n-1} = \omega_0 + K(1 - \omega_0)\left(\frac{1}{2} + \dots + \frac{1}{p}\right) = 1$$

whence

(518-8) $$K = \frac{1}{\frac{1}{2} + \dots + \frac{1}{p}}$$

and

(518-9) $$\Sigma n\, \omega_{n-1} = \omega_0 + \frac{(1 - \omega_0)(p-1)}{\frac{1}{2} + \dots + \frac{1}{p}}$$

We will have therefore for any values of ω_0 and p.

(518-10) $$\Delta < \frac{1}{2}\,\frac{p+1}{(p-1)}\,\frac{\frac{1}{2} + \dots + \frac{1}{p}}{1 - \omega_0}$$

so that for instance in the case of hypothesis (518-2)

(518-11) $$\Delta < \frac{1}{1 - \omega_0} \quad \text{for} \quad p = 5$$

Thus we see that if ω_o is not too large, Δ will not differ greatly from 1.

On the contrary, if ω_o is near unity the superior limit of Δ given by (518-10) becomes very large. For the values of ω_o near unity the inequality (518-5) becomes

(518-12) $$\Delta < \frac{p+1}{2\,\omega_o}$$

It may at first sight appear to be a rather surprising result that if ω_o is near unity, that is, if the relative weight given to the term $\beta_o(\theta)$ corresponding to the exponential model [1] is relatively large, Δ can reach very high values whereas in the case of the exponential model, $\Delta = 1$ [2].

---

[1] Relation (510-13).

[2] From (518-6) and (518-2)

(1) $$\Sigma\; n\; \omega_{n-1} = \omega_o + k\;(p-1)\;(1-\omega_o)$$

(2) $$\Sigma\; n^2 \omega_{n-1} = \omega_o + k\;\frac{(p-1)\;(p+2)}{2}\;(1-\omega_o)\;.$$

Thus from (512-14)

(3) $$\Delta = \frac{\Sigma\; n\; \omega_n + \Sigma\; n^2\; \omega_{n-1}}{2\;(\Sigma\; n\; \omega_{n-1})^2}$$

and

(4) $$\Delta = \frac{4\;\omega_o + k\;(p-1)\;(p+4)\;(1-\omega_o)}{4\;[\omega_o + k\;(p-1)\;(1-\omega_o)]^2}$$

For small values of ω_o

(5) $$\Delta \backsim 1\;.$$

For small valves of ω_o

6) $$\Delta \sim \frac{p+4}{4\,(p-1)}\left[\frac{1}{2} + \ldots + \frac{1}{p}\right]$$

If $p = 5$, $\Delta \backsim 0.72$.

This finding will be verified later for a particular case [1].

These results can be summarised in the following table. On the whole, we see that in the majority of cases which can reasonably be envisaged, Δ does not depart greatly from unity. Since in all cases the integrals

$$\begin{cases} k = \int_0^\infty \beta(\theta)\, d\theta \\ \Theta_0 = \frac{1}{k} \int_0^\infty \theta\, \beta(\theta)\, d\theta \end{cases}$$

have values which are near 1 and γ_0 respectively [2], and since

$$\Delta = \frac{1}{2k\Theta_0^2} \int_0^\infty \theta^2\, \beta(\theta)\, d\theta \qquad [3]$$

it follows that in the majority of cases which can reasonably be considered, the first three moments of the function $\beta(\theta)$ are near to the corresponding moments of the exponential model. It then follows that the related properties differ little from those of the exponential model.

If p is very large

$$(7) \qquad \Delta \sim \frac{1}{4}\left(\frac{1}{2} + \ldots + \frac{1}{p}\right)\frac{1}{1-\omega_0}$$

$$\sim \frac{Lp}{4}\,\frac{1}{1-\omega_0}$$

Thus if p increases indefinitely and if $1-\omega_0$ tends to unity, Δ increases indefinitely. However, for $\omega_0 = 1$ the model reduces to the exponential model and $\Delta = 1$ (see below Chart IV which corresponds to a particular case).

(1) § 528 to 532 and § 540.

(2) § 311 and 313.

(3) Relation (220-6).

TABLE 14

	ω_{n-1}	Δ
Infinite series	$\omega_{n-1} = \frac{\lambda}{n^q}$	$1 < \Delta < 1,06$
	$\omega_{n-1} = \lambda e^{-K(n-1)}$	$\Delta = 1$
	$\omega_{n-1} = \lambda n^q e^{-K(n-1)}$ $q = 1$	$\frac{3}{4} < \Delta < 1$
Finite series of p terms $\omega_0, \omega_1, \ldots, \omega_{p-1}$	$\omega_q = \frac{1}{p}$	$\frac{2}{3} < \Delta \leqslant 1$
	$\frac{\Sigma n \omega_{n-1}}{\Sigma \omega_{n-1}} > \frac{p}{2}$	$\frac{1}{2} < \Delta < 1 + \frac{1}{p}$
	$n \omega_{n-1}$ never increasing	$\Delta < \frac{1}{2} \frac{p+1}{\Sigma n \omega_{n-1}}$
	$n \omega_{n-1} = K(1-\omega_0)$ $n \geqslant 2$	$\Delta < \frac{1}{2} \frac{p+1}{p-1} \frac{\frac{1}{2} + \cdots + \frac{1}{p}}{1-\omega_0}$

[11] *Allais* - pag. 230

It may be recalled that according as Δ is below or above unity, amortisation is more or less rapid than in the exponential model [1].

Value of Θ

519. From the limited expansion (314-6) we have

(519-1) $$\Theta - \gamma_C \sim - \delta \gamma_G^2 (i - \rho)$$

If it be assumed that

(519-2) $$|\delta| < 1/2$$

a condition which is met in the majority of cases, then

(519-3) $$\gamma_C \left[1 - \frac{\gamma_C}{2}(i - \rho)\right] < \Theta < \gamma_C \left[1 + \frac{\gamma_C}{2}(i - \rho)\right]$$

To see this more specifically, consider the case of the United States for which

(519-4) $$\gamma = 3.4 \qquad i = 4\% \qquad \rho = 1.7\%$$

(1) § 336.

Now

(519-5) $$\gamma_c = \frac{\gamma}{1 - \gamma\rho}$$ (1)

so that

(519-6) $$\gamma_c = 3.61 \qquad \frac{\gamma_c}{2}(i - \rho) = 0.04$$

Thus we see that under the assumptions made, Θ differs little from γ_c.

In any case we have from (123-12)

$$\Theta < \frac{\gamma_c}{1 - (i - \rho)\,\gamma_c}$$

whatever the function $\varphi(\theta)$.

The above results are illustrated in the following study of two particular cases.

(1) Main text, relation (125-4).

II. — PARTICULAR CASES

A. The development of $\beta(\theta)e^{\mu\theta}$ reduces to the n^{th} term ($n > 2$) (para-exponential model)

Hypotheses

520. With the exception of the exponential model ($n = 1$) the simplest variant of the general case which has just been studied corresponds to the equalities

$$\beta(\theta) = \frac{b\theta^{n-1}}{\overline{n-1}!} e^{-\mu\theta} \qquad n \geqslant 2 \tag{520-1}$$

$$k = 1 \tag{520-2}$$

For want of a better term, this will be denoted as the para-exponential model (1).

Expressions for the main quantities

521. We have

$$b = \mu^n \tag{521-1}$$

(1) For $n = 1$ these formulae correspond, of course, to the exponential model (§ 250).

whence according to (220-5) and (220-6)

(521-2) $$\Theta_0 = \frac{n}{\mu}$$

(521-3) $$\Delta = \frac{1}{2}\left(1 + \frac{1}{n}\right)$$

Consequently, applying the general formulae ([1]), we find

(521-4) $$\beta(\theta) = \frac{n^n}{\overline{n-1}!}\left(\frac{\theta}{\Theta_0}\right)^{n-1} \frac{e^{-n\frac{\theta}{\Theta_0}}}{\Theta_0}$$

(521-5) $$\psi(u) = \frac{1}{\left[1 + \frac{\Theta_0}{n} u\right]^n}$$

(521-6) $$\varphi(\theta) = \frac{n^n}{\overline{n-1}!}\left(\frac{\theta}{\Theta}\right)^{n-1} \frac{e^{-n\frac{\theta}{\Theta_0}}}{\Theta}$$

(521-7) $$\Theta = \frac{\Theta_0}{1 + \frac{\Theta_0}{n}(i-\rho)}$$

(521-8) $$\frac{R_C}{R_\omega} = \left[1 + \frac{\Theta_0}{n}(i-\rho)\right]^n$$

(521-9) $$\gamma_C = \frac{1}{(i-\rho)}\left[1 - \frac{1}{\left[1 + \frac{\Theta_0}{n}(i-\rho)\right]^n}\right]$$

(521-10) $$\frac{\overline{R}_C}{\overline{R}_{CM}} = \left[1 + \frac{\Theta_0}{n}(i-\rho)\right]^n e^{-\Theta_0(i-\rho)}$$

([1]) Main text, relations (220-1), (220-5), (223-7), (227-3), (224-2), (226-2) and (228-12).

All these expressions remain *unchanged* if, with n being any positive number whatever (not necessarily an integer), we have

(521-11) $$\beta(\theta) = b\,\frac{\theta^{n-1}}{\Gamma(n)}\,e^{-\mu\theta}$$

where $\Gamma(n)$ is the function Gamma (1) (2).

The Value of the Functions $\beta(\theta)$ *and* $\varphi(\theta)$

522. For given Θ and small $i-\rho$, Θ differs little from Θ_o, and from relation (521-2) we see that μ is greater for larger values of n. It follows that at infinity, for given θ, $\beta(\theta)$ and $\varphi(\theta)$ take smaller values as n becomes larger. The effect of increasing n is therefore to concentrate the masses $\varphi(\theta)d\theta$ around the average values Θ.

---

(1) For we have

$$k = \int_0^\infty \beta(\theta)\,d\theta = \frac{b}{\mu^n}$$

$$\Theta_0 = \frac{1}{k}\int_0^\infty \theta\,\beta(\theta)\,d\theta = \frac{1}{k}\,\frac{\Gamma(n+1)}{\Gamma(n)}\,\frac{b}{\mu^{n+1}} = \frac{n\,b}{k\mu^{n+1}} = \frac{n}{\mu}$$

$$\psi(u) = \frac{1}{k}\int_0^\infty \beta(\theta)\,e^{-u\theta}\,d\theta = \frac{1}{k}\,\frac{b}{(\mu+u)^n} = \frac{1}{\left(1+\frac{\Theta_0}{n}\,u\right)^n}$$

(2) Naturally $\varphi(\theta)$ becomes

$$\varphi(\theta) = \frac{n^n}{\Gamma(n)}\left(\frac{\theta}{\Theta}\right)^{n-1}\frac{e^{-n\frac{\theta}{\Theta_0}}}{\Theta}$$

The function $\varphi(\theta)$ is zero for $\theta=0$ and $\theta=\infty$, and its *maximum* is reached for

(522-1)
$$\theta=\theta_m=\frac{n-1}{n}\,\frac{\Theta_0}{1+\frac{\Theta_0}{n}(i-\rho)}=\frac{n-1}{n}\,\Theta$$
$$=\frac{n-1}{\mu+u}=\frac{(n-1)T}{1+Tu}$$

For infinite n we have

(522-2)
$$\theta_m=\Theta_0$$

and

(522-3)
$$\Theta=\Theta_0 \qquad \Delta=\frac{1}{2}$$

The whole mass of the $\beta(\theta)d\theta$ is concentrated on the point of the abscissa Θ_0 (1).

The influence of n

523. Assuming that, in conformity with the statistical data,

(523-1)
$$\gamma_C=3.5 \qquad i-\rho=4\%$$

(1) Since $\Delta=1/2$ (see main text, § 220).

the comparative values of the most interesting quantities are summarised in the following table for $n = 1, 2, 3$ and ∞ (1) (2).

$\gamma_c = 3.5 \quad i-\rho = 4\%$	n			
Magnitudes	1	2	3	∞
Θ_0	4.06	3.92	3.87	3.77
Δ	1	0.75	0.67	0.5
Θ	3.50	3.63	3.68	3.77
$\dfrac{\Theta-\gamma_c}{\gamma_c}$	0	0.04	0.05	0.08
$\int_0^1 \varphi(\theta)\, d\theta$	0.25	0.11	0.05	0
$\int_1^5 \varphi(\theta)\, d\theta$	0.51	0.65	0.72	1
$\int_5^\infty \varphi(\theta)\, d\theta$	0.24	0.24	0.23	0
R_c/R_ω (3)	1.163	1.163	1.163	1.163
$\dfrac{\overline{R}_{cM}-\overline{R}_c}{\overline{R}_{cM}}$	1.2%	0.6%	0.4%	0

(1) For the expression (521-6) of $\varphi(\theta)$ we have

$$\int_a^b \varphi(\theta)\, d\theta = e^{-\frac{a}{\Theta}} - e^{-\frac{b}{\Theta}} \qquad \text{for } n = 1$$

$$\int_a^b \varphi(\theta)\, d\theta = e^{-\frac{2a}{\Theta}}\left(1+\frac{2a}{\Theta}\right) - e^{-\frac{2b}{\Theta}}\left(1+\frac{2b}{\Theta}\right) \qquad \text{for } n = 2$$

$$\int_a^b \varphi(\theta)\, d\theta = \left[\frac{9a^2}{2\Theta^2}+\frac{3a}{\Theta}+1\right] e^{-\frac{3a}{\Theta}} - \left[\frac{9b^2}{2\Theta^2}+\frac{3b}{\Theta}+1\right] e^{-\frac{3b}{\Theta}} \qquad \text{for } n = 3$$

(2) For $n = \infty$ the method of obtaining the limits is given later (relations 523-2 to 523-9).

(3) Variations in this quantity corresponding to variations in n are completely negligible.

This table shows clearly the influence of n on the results.

It can be seen that the average amortization period Θ is in all cases very near to the value of γ_C. The value of Θ_0 remains practically unchanged as is also the case for R_C/R_ω. The loss of real income $(\bar{R}_{CM} - \bar{R}_C)/\bar{R}_{CM}$ is very low in all cases, and furthermore the value of this expression declines as n increases.

The only quantities to vary appreciably correspond to the amortisation schedule, as can be seen from the values of the integral $\int_0^1$, $\int_1^5$ and $\int_5^\infty$ of $\varphi(\theta)$.

Charts I, II and III represent the variations of the functions

$$\beta(\theta)\,,\quad \varphi(\theta)\,,\quad \int_\theta^\infty \varphi(\theta)\,d\theta$$

for n equal to 1, 2, 3 and 5. It will be seen that the elasticity $\beta(\theta)$ declines more rapidly for higher values of n. Similarly, amortisation is the more rapid for high n, but while the differences are real ones, they are not substantial.

Very High Value of n

We can have a good idea of what happens when n takes on high values by considering what the formulae already given become when n is infinitely large. We have then [1]

(523-2) $$\psi(u) = e^{\Theta_0 u} \qquad \Delta = \frac{1}{2}$$

(523-3) $$\Theta = \Theta_0$$

(1) From § 521.

$$\frac{R_C}{R_\omega} = e^{\Theta_0 (i-\rho)} \tag{523-4}$$

$$\begin{cases} \gamma_C = \dfrac{1}{(i-\rho)}\left[1 - e^{-\Theta_0 (i-\rho)}\right] & \text{(523-5)} \\ \gamma_C \sim \Theta_0 \left[1 - \dfrac{\Theta_0}{2}(i-\rho)\right] & \text{(523-6)} \end{cases}$$

$$\frac{\overline{R}_{CM} - \overline{R}_C}{\overline{R}_{CM}} = 0 \tag{523-7}$$

Further, from (521-4), (521-6) and Stirling's formula

$$\beta(\theta) \sim \sqrt{\frac{n}{2\pi}} \left(\frac{\theta}{\Theta_0}\right)^{n-1} e^{-n\left(\frac{\theta}{\Theta_0} - 1\right)} \tag{523-8}$$

$$\varphi(\theta) \sim \sqrt{\frac{n}{2\pi}} \left(\frac{\theta}{\Theta_0}\right)^{n-1} e^{-n\left(\frac{\theta}{\Theta_0} - 1\right)} e^{(\Theta_0 - \theta)(i-\rho)} \tag{523-9}$$

It can be verified without difficulty that for n infinite, $\rho(\theta)$ and $\varphi(\theta)$ are infinite for $\theta = \Theta_o$ and zero for any other value of θ. The whole mass of the $\beta(\theta) d\theta$ is concentrated on the point $\theta = \Theta_o$. When γ_C and $i - \rho$ are given by observation, Θ_o is determined from equation (523-5).

CHART I

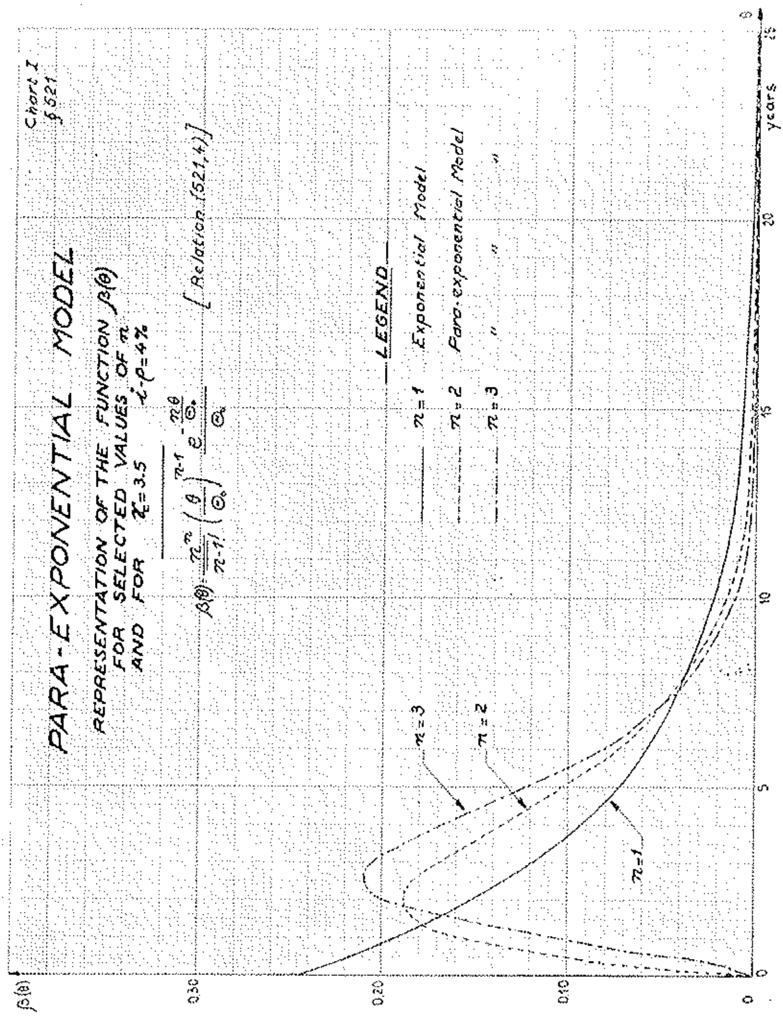

PARA - EXPONENTIAL MODEL
REPRESENTATION OF THE FUNCTION β(θ)
FOR SELECTED VALUES OF π
AND FOR Χ = 3.5 i - ρ = 4%
[Relation (521,4)]
Chart I
§ 521
LEGEND
π = 1 Exponential Model
π = 2 Para-exponential Model
π = 3 " "
π=1
π=2
π=3
β(θ)
0.30
0.20
0.10
0
0
5
10
15
20
25
years

CHART II

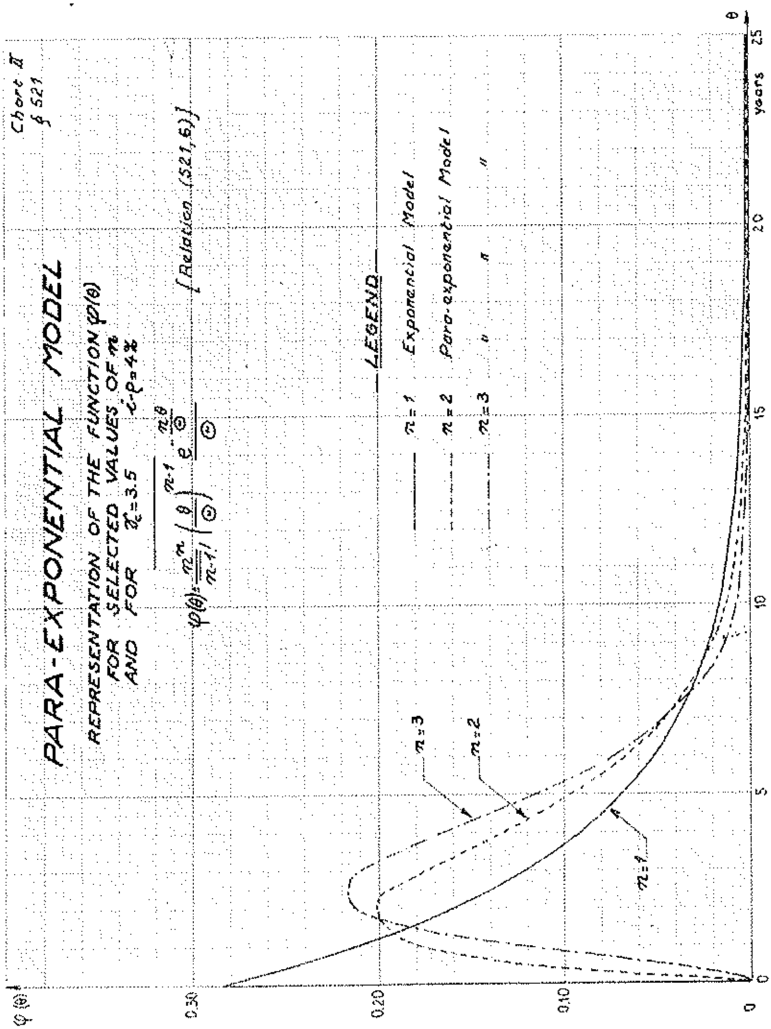


Chart II
§ 521
PARA-EXPONENTIAL MODEL
REPRESENTATION OF THE FUNCTION φ(θ)
FOR SELECTED VALUES OF n
AND FOR n_c=3.5 i-ρ=4%
$\varphi(\theta)=\frac{n^n}{n-1!}\left(\frac{\theta}{\Theta}\right)^{n-1}\frac{e^{-\frac{n\theta}{\Theta}}}{\Theta}$
[Relation (521,6)]
LEGEND
n=1 Exponential Model
n=2 Para-exponential Model
n=3 " "
n=3
n=2
n=1
φ(θ)
0.30
0.20
0.10
0
0
5
10
15
20
25
years
θ

CHART III

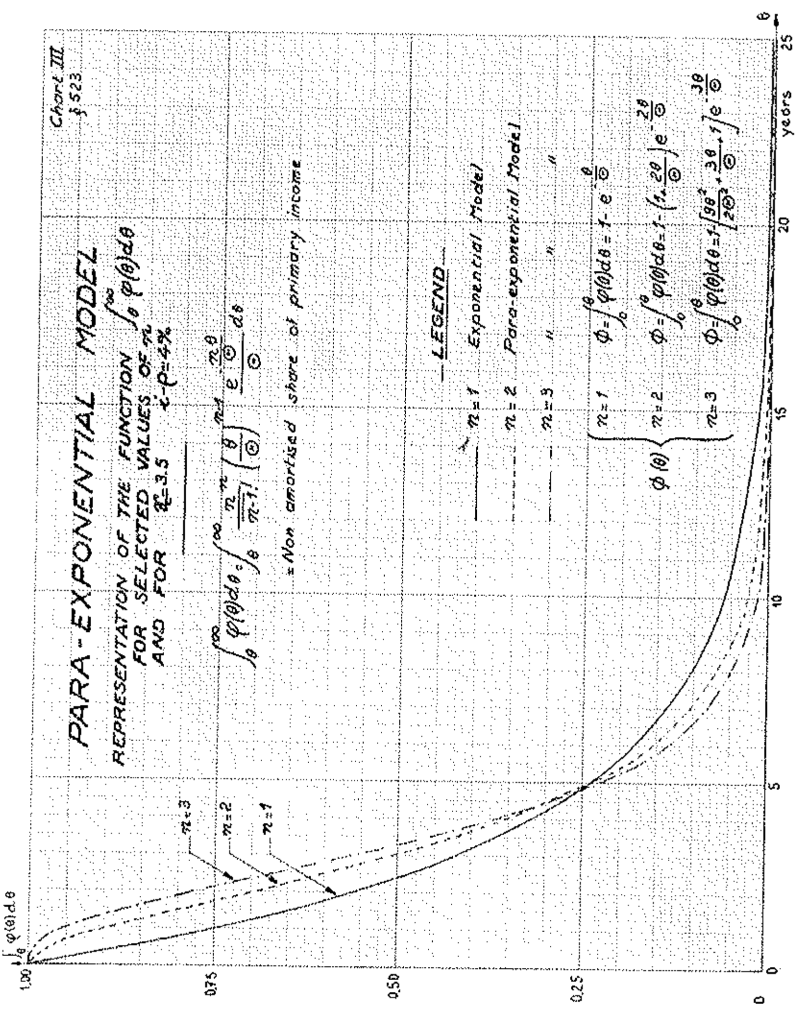


Chart III
§ 523
PARA - EXPONENTIAL MODEL
REPRESENTATION OF THE FUNCTION
FOR SELECTED VALUES OF n
AND FOR
× Non amortised share of primary income
LEGEND
n = 1 Exponential Model
n = 2 Para-exponential Model
n = 3
n=3
n=2
n=1
years
1.00
0.75
0.50
0.25
0
0
5
10
15
20
25

Determination of Θ_0 *and* n

524. In the present case, to determine the constants Θ_0 and n, we have the two relations

$$\begin{cases} \Theta = \dfrac{\Theta_0}{1 + \dfrac{\Theta_0}{n}(i-\rho)} & (524\text{-}1) \\[2ex] \gamma_c = \dfrac{1}{i-\rho}\left[1 - \dfrac{1}{\left[1 + \dfrac{\Theta_0}{n}(i-\rho)\right]^n}\right] & (524\text{-}2) \end{cases}$$

whence

$$\begin{cases} \Theta_0 = \dfrac{1}{\dfrac{1}{\Theta} - \dfrac{i-\rho}{n}} & (524\text{-}3) \\[2ex] \gamma_c = \dfrac{1}{i-\rho}\left[1 - \dfrac{1}{\left[1 + \dfrac{i-\rho}{\dfrac{n}{\Theta} - (i-\rho)}\right]^n}\right] & (524\text{-}4) \end{cases}$$

and therefore

$$\left[1 - \frac{\Theta}{n}(i-\rho)\right]^n = 1 - \gamma_c(i-\rho) \qquad (524\text{-}5)$$

an implicit equation which determines n as a function of γ_c and Θ.

It can easily be verified that even in the very simple case where $\beta(\theta)$ *is assumed to be of the form* (520-1), *the determination of* Θ_0 *and* n *is subject to a very large margin of error if the value of* γ_c *is near that of* Θ *and* Θ *itself is not very accurately known.*

The following table gives the values of Θ as a function of n for $\gamma_c = 3.5$ and $i - \rho = 4\%$ (1).

The relation between n and Θ for $\gamma_c = 3.5$ and $i - \rho = 4\%$

n	1	2	3	5	∞
Θ	3.50	3.63	3.68	3.70	3.77

This table shows that the slightest variation in Θ involves a very large change in n.

The fact that Θ be given is thus an insufficient basis for determining the constants of the model. The function $\varphi(0)$ must be considered as a whole.

(1) For n infinite, relation (523-5) gives

$$e^{-\Theta(i-\rho)} = 1 - \gamma_c (i - \rho)$$

B. THE EXPANSION OF $\beta(\theta)e^{\mu\theta}$ REDUCES TO THE FIRST AND n^{th} TERMS (MIXED MODEL)

Hypotheses

525. Assume now that

$$(525\text{-}1)\qquad \begin{cases} \beta(\theta) = \left[a + \dfrac{b}{\overline{n-1}!}\,\theta^{\,n-1}\right] e^{-\mu\theta} \\ \mathrm{K} = 1 \end{cases}$$

(525-2)

Since $\beta(\theta)$ must be positive whatever θ, and is zero at infinity, we can write

$$(525\text{-}3)\qquad a > 0 \qquad b > 0 \qquad \mu = \frac{1}{\mathrm{T}} > 0$$

Expressions for the Main Quantities

526. Put

$$(526\text{-}1)\qquad \omega = \frac{b}{\mu^{n}} \qquad \omega' = \frac{a}{\mu}$$

Then, from (510-3) and (525-2)

(526-2) $$\omega + \omega' = 1$$

and from (510-6)

(526-3) $$\beta(\theta) = (1 - \omega)\beta_o(\theta) + \omega\beta_{n-1}(\theta)$$ (¹)

According to (512-6), (512-7), (510-6), (510-7), (511-1), (511-3), (227-3), (224-2), (226-2) and (228-12), we have then

(526-4) $$\Theta_0 = \left[1-\omega + \omega n\right] T$$

(526-5) $$\Delta = \frac{1-\omega+\omega \dfrac{n(n+1)}{2}}{\left[1-\omega+\omega n\right]^2}$$

(526-6) $$\beta(\theta) = \left[1-\omega+\omega \frac{1}{\overline{n-1}!}\left(\frac{\theta}{T}\right)^{n-1}\right] \frac{e^{-\frac{\theta}{T}}}{T}$$

(526-7) $$\psi(u) = \frac{1-\omega}{1+Tu} - \frac{\omega}{\left[1+Tu\right]^n}$$

(526-8) $$\varphi(\theta) = \frac{\left[1-\omega+\omega \dfrac{1}{\overline{n-1}!}\left(\dfrac{\theta}{T}\right)^{n-1}\right]}{\dfrac{1-\omega}{1+T(i-\rho)} - \dfrac{\omega}{\left[1+T(i-\rho)\right]^n}} \; \frac{e^{-\left[1+T(i-\rho)\right]\frac{\theta}{T}}}{T}$$

(¹) The present notation thus corresponds to that used in the previous section [relation (510-9)] as follows

$$\omega = \omega_{n-1}$$
$$\omega' = 1 - \omega = \omega_o \, .$$

(526-9) $$\Theta = \frac{1-\omega + \dfrac{\omega n}{[1+T(i-\rho)]^{n-1}}}{1-\omega + \dfrac{\omega}{[1+T(i-\rho)]^{n-1}}} \; \frac{T}{1+T(i-\rho)}$$

(526-10) $$\frac{R_C}{R_\omega} = \frac{1}{\dfrac{1-\omega}{1+T(i-\rho)} + \dfrac{\omega}{[1+T(i-\rho)]^n}}$$

(526-11) $$\Upsilon_C = \frac{1}{(i-\rho)}\left[1 - \frac{1-\omega}{1+T(i-\rho)} - \frac{\omega}{[1+T(i-\rho)]^n}\right]$$

(526-12) $$\frac{\overline{R}_C}{\overline{R}_{CM}} = \frac{1}{\dfrac{1-\omega}{1+T(i-\rho)} + \dfrac{\omega}{[1+T(i-\rho)]^n}} \, e^{-\Theta_0 (i-\rho)}$$

It is of course also possible to write

(526-13) $$\varphi(\theta) = \frac{\beta(\theta) e^{-(i-\rho)\theta}}{1-\Upsilon_C(i-\rho)}$$

(526-14) $$\Theta = \frac{\dfrac{1-\omega}{1+T(i-\rho)} + \dfrac{\omega n}{[1+T(i-\rho)]^n}}{1-\Upsilon_C(i-\rho)} \; \frac{T}{1+T(i-\rho)}$$

(526-15) $$\frac{R_C}{R_\omega} = \frac{1}{1-\Upsilon_C(i-\rho)}$$ [1]

[1] Naturally we find the general relation (226-9).

$$(526\text{-}16) \qquad \frac{\overline{\overline{R}}_C}{\overline{\overline{R}}_{CM}} = \frac{e^{-\Theta_0(i-\rho)}}{1-\Upsilon_C(i-\rho)} \qquad (^1)$$

These relations are more convenient for use where γ_C is given.

When $\omega = 0$, *this model reduces to the exponential model of the main text and when* $\omega = 1$, *it reduces to the para-exponential model examined in the preceding section* (*with* $n \geqslant 2$).

The case $n = 2$ is particularly important. In this case, the Taylor expansion of $\beta(\theta)e^{\mu\theta}$ can be limited to the first two terms.

The Form of $\varphi(\theta)$

527. $\varphi(\theta)$ is maximum for $\theta = \theta_m$ such that

$$(527\text{-}1) \qquad \frac{\theta_m}{T} = \frac{n-1}{[1+T(i-\rho)]}\left[1-\overline{n-2}\,!\,\frac{1-\omega}{\omega}[1+T(i-\rho)]\left(\frac{T}{\theta_m}\right)^{n-2}\right]$$

In all cases we have

$$(527\text{-}2) \qquad \frac{\theta_m}{T} < \frac{n-1}{[1+T(i-\rho)]}$$

(1) Again we find the general relation (228-14) for $h = 1$.

and there is no longer a maximum value for

$$(527\text{-}3) \qquad \left(\frac{\theta}{T}\right)^{n-2} < \overline{n-2}\,!\,\frac{1-\omega}{\omega}\left[1+T(i-\rho)\right]$$

θ_m decreases when ω decreases, i.e. as the exponential model is approached.

Values of Δ

528. Although the model reduces to the exponential model $[b=0,\ \Delta=1]$ for $\omega=0$ (1), and to the para-exponential model $[a=0]$ with

$$\Delta = \frac{1}{2}\left(1+\frac{1}{n}\right)$$

for $\omega=1$, *it would be incorrect to deduce* that in all cases the properties of this model always lie somewhere between those of the exponential and para-exponential models, and in particular that we would always have

$$(528\text{-}1) \qquad \frac{1}{2} < \Delta < 1 .$$

For, from (526-5), however small ω, there exist values of n which are sufficiently large for $n\omega$ greatly to exceed $1-\omega$.

(1) Or for $b \neq 0$ with $n=1$.

For these values, we have

(528-2) $$\Delta \sim \frac{1}{2\omega}$$

and $1/2\omega$ can of course be very large when ω is very small (1).

First Limitation of the Value of Δ

529. Providing ω is not too small, Δ will remain at a relatively low level. Indeed, it may be noted in the first place that

(1) We have

$$\frac{\Delta}{\frac{1}{2\omega}} = \left(1 + \frac{1}{n}\right) \frac{\left[1 + \frac{2(1-\omega)}{n(n+1)\omega}\right]}{\left[1 + \frac{1-\omega}{n\omega}\right]^2}$$

Thus Δ will differ from its limit value by less than ε, i.e. we will have

$$\frac{\Delta}{\frac{1}{2\omega}} < 1 + \varepsilon$$

if

$$\frac{1-\omega}{n\omega} < \frac{\varepsilon}{3}$$

i.e. if

$$n > \frac{3(1-\omega)}{\omega\varepsilon}$$

Thus, if for example $\omega = 1/100$, Δ will differ from its limit value 50 by less than $1/10$ for $n > 3000$. It can thus be seen that Δ can only have very high values if n is very large, and such values are incompatible with observed reality, at least as far as can be judged.

from condition (517-7) we have [1]

$$(529\text{-}1) \qquad \frac{1}{2} < \Delta < 1 + \frac{1}{n} \qquad \text{for} \qquad \omega > 1 - \frac{1}{2}\,\frac{n}{n-1}$$

Second Limitation on the value of Δ

530. Moreover, since we have according to (526-5)

$$(530\text{-}1) \qquad \frac{d\Delta}{dn} = \frac{1}{2}\,\frac{\omega\left[(2-3\,\omega)\,n-3\,(1-\omega)\right]}{\left[1-\omega+\omega\,n\right]^3}$$

we see that there are then three cases:

I: $0 < \omega < 1/3$

n	2		∞
Δ	$\dfrac{1+2\omega}{(1+\omega)^2}$	$\nearrow$	$\dfrac{1}{2\,\omega}$

here we have

$$(530\text{-}2) \qquad \frac{15}{16} < \Delta < \infty \qquad \text{for} \qquad 0 < \omega < \frac{1}{3}$$

---

(1) For here condition (517-6) can be written

$$1 - \omega + n\omega > \frac{n}{2}$$

II: $1/3 < \omega < 2/3$

$$\begin{array}{c|cccc|}
n & 2 & & n' & \infty \\
\Delta & \dfrac{1+2\omega}{(1+\omega)^2} & \searrow \ \Delta' & \nearrow & \dfrac{1}{2\omega}
\end{array}$$

So that

(530-3) $$\frac{1}{2} < \Delta < \frac{3}{2} \qquad \text{for} \qquad \frac{1}{3} < \omega < \frac{2}{3}$$

III: $2/3 < \omega < 1$

$$\begin{array}{c|ccc|}
n & 2 & & \infty \\
\Delta & \dfrac{1+2\omega}{(1+\omega)^2} & \searrow & \dfrac{1}{2\omega}
\end{array}$$

Thus we have

(530-4) $$\frac{1}{2} < \Delta < \frac{21}{25} \qquad \text{for} \qquad \frac{2}{3} < \omega < 1$$

It follows from this that *whatever the value of* n we have

(530-5) $$\frac{1}{2} < \Delta < \frac{3}{2} \qquad \text{for} \qquad \frac{1}{3} < \omega < 1$$

whence

$$(530\text{-}6) \qquad -\frac{1}{2} < \delta < \frac{1}{2} \qquad \text{for} \qquad \frac{1}{3} < \omega < 1 \qquad (^1)$$

Third Limitation of the Value of Δ

531. From (526-4) we have

$$(531\text{-}1) \qquad \omega = \frac{\mu\,\Theta_0 - 1}{n-1}$$

and we can write

$$(531\text{-}2) \qquad \Delta = \frac{1}{1-\omega+n\,\omega} + \frac{n(n-1)}{2}\,\frac{\omega}{\left[1-\omega+n\,\omega\right]^2}$$

so that we have

$$(531\text{-}3) \qquad \Delta = v + \frac{n(n-1)}{2}\left(1 - \frac{n - \frac{1}{v}}{n-1}\right) v^2$$

---

(1) These results show that ω has to be large enough for Δ to remain sufficiently small. This is the general result given in § 518 of the Appendix. It may be recalled that in all cases

$$1/2 < \Delta \qquad\qquad \omega < 1$$

conditions (220-13) and (526-2).

i.e.

$$\Delta = \frac{n+2}{2}\, v - \frac{n}{2}\, v^2 \tag{531-4}$$

putting

$$v = \frac{1}{\mu \Theta_0} \tag{531-5}$$

The discussion can be summarised in the following table

v	0	v_1	$\frac{n+2}{2n}$	v_2	$\frac{n+2}{n}$	$+\infty$
Δ	$0 \nearrow$	$\frac{1}{2} \nearrow$	Δ_m	$\searrow \frac{1}{2}$	$\searrow 0$	$\searrow -\infty$

where v_1 and v_2 are the values of v for which $\Delta = 1/2$ and where

$$\Delta_m = \frac{(n+2)^2}{8n} \tag{531-6}$$

We verify $\Delta_m > 1/2$ whatever the value of n (¹).
Thus *in every case we have*

$$\Delta < \frac{(n+2)^2}{8n} \tag{531-7}$$

(¹) Condition (220-13).

For selected values of n, the values of Δ_m are as follows

n	2	3	4	5	6	7	8
Δ_m	1	1.04	1.12	1.22	1.33	1.45	1.56

Thus we see that when the value of n is not too large, Δ remains fairly near 1.

In any case

(531-8) $\quad \Delta < 3/2 \quad$ for $\quad n \leqslant 7$

so that

(531-9) $\quad \delta < 1/2 \quad$ for $\quad n \leqslant 7$

whatever the value of ω.

Recapitulation

532. The foregoing results are summarised in the following table

$$\frac{1}{2} < \Delta < 1 + \frac{1}{n} \quad \text{for} \quad \omega > 1 - \frac{1}{2}\,\frac{n}{n-1}$$

$$\frac{1}{2} < \Delta < \frac{3}{2} \quad \text{for} \quad \frac{1}{3} < \omega < 1$$

$$\frac{1}{2} < \Delta < \frac{3}{2} \quad \text{for} \quad 1 \leqslant n \leqslant 7$$

Finally, it should be remarked that even for $\omega < 1/3$ Δ differs significantly from unity only for very small values of ω and very large values of n. In fact, we have the following table (1) (2):

Values of Δ							
n \ ω	0	0.05	0.10	0.25	0.50	0.75	1
1	1	1	1	1	1	1	1
2	1	1.00	0.99	0.96	0.89	0.82	0.75
3	1	1.03	1.04	1.00	0.87	0.76	0.67
5	1	1.18	1.22	1.12	0.89	0.72	0.60
10	1	1.76	1.77	1.37	0.93	0.69	0.55
∞	1	10	5	2	1	0.67	0.5

For the particularly important case, n = 2, we have

$$\Delta = \frac{1+2\omega}{(1+\omega)^2}$$

whence

$$3/4 < \Delta < 1$$

It will be recalled that

$$\Delta = \frac{1}{2\omega} \qquad \text{for} \quad n = \infty \qquad (3)$$

(1) Relation (526-5).
(2) The values of Δ for $n = \infty$ are given only for indicative purposes. It goes without saying that very large values of n are completely unrealistic.
(3) Relation (528-2).

CHART IV

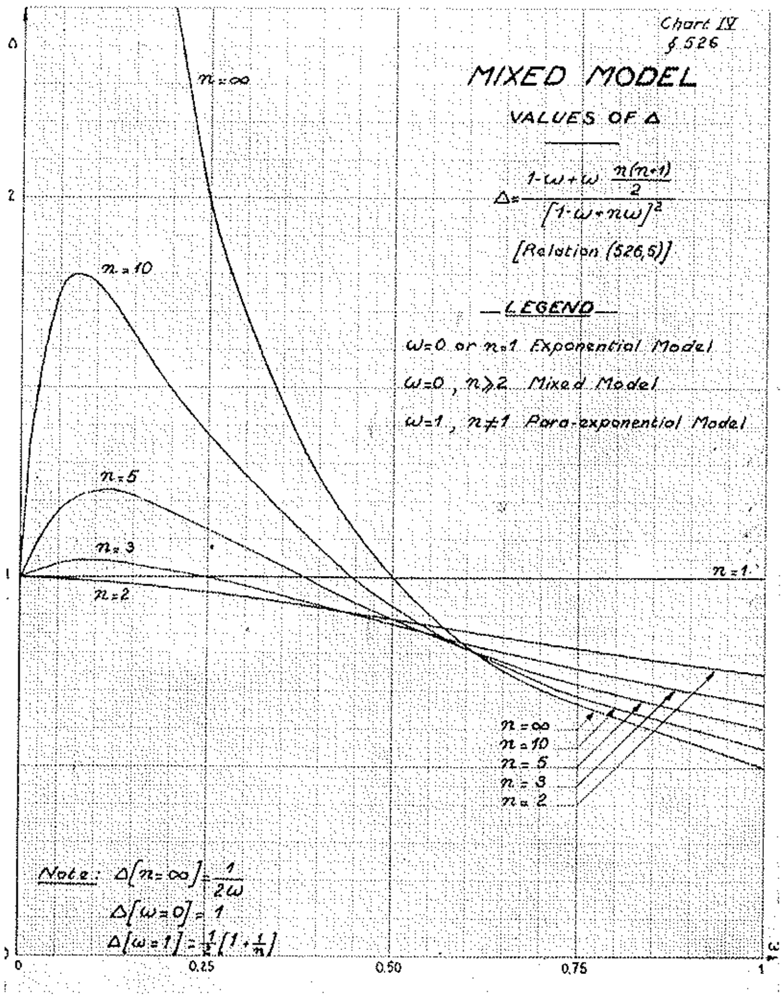


Chart IV
§ 526
MIXED MODEL
VALUES OF Δ
$\Delta = \frac{1-\omega+\omega\frac{n(n+1)}{2}}{[1-\omega+n\omega]^2}$
[Relation (526,5)]
— LEGEND —
ω=0 or n=1 Exponential Model
ω=0 , n≥2 Mixed Model
ω=1 , n≠1 Para-exponential Model
n=∞
n=10
n=5
n=3
n=2
n=1
Note: $\Delta[n=\infty]=\frac{1}{2\omega}$
$\Delta[\omega=0]=1$
$\Delta[\omega=1]=\frac{1}{2}[1+\frac{1}{n}]$
0
0.25
0.50
0.75
1
ω
2
1

Chart IV represents the variation of Δ with n for different values of ω. The reader is reminded at this point that it follows from the discussion in § 336 that amortization will be more or less rapid than in the exponential model according as Δ is less than or greater than unity.

Thus we see that over a wide range of variation of ω, the properties of the model considered are near those of the exponential model.

Analogous calculations could of course be undertaken for the general case discussed in § 510. *They would show that in the majority of cases which can reasonably be envisaged, the value of Δ remains fairly near unity.*

Determination of μ

533. In any case, it should be clearly understood that when the value of γ_C is given, the range of variation of $\mu = 1/T$ is limited.

For we have [1]

$$\gamma_C = \frac{1}{(i-\rho)}\left[1 - \frac{1-\omega}{1+T(i-\rho)} - \frac{\omega}{[1+T(i-\rho)]^n}\right] \tag{533-1}$$

To simplify the discussion, take

$$i - \rho = 0 \tag{533-2}$$

which can be considered as a valid first approximation since $i - \rho$ *is* small. Then

$$\gamma_C(i-\rho) = \Theta_0 \tag{533-3}$$

(1) Relation (526-11).

and since necessarily

(533-4) $$\Delta > \frac{1}{2}$$ [1]

it follows from the discussion in § 531 that we should have

(533-5) $$v_1 < v < v_2$$

i.e.

(533-6) $$v_1 \Theta_o < T < v_2 \Theta_o$$

with

(533-7) $$v_1 = \frac{1}{2} + \frac{1}{n} - \frac{1}{2}\sqrt{1 + \frac{4}{n^2}} \qquad v_2 = \frac{1}{2} + \frac{1}{n} + \frac{1}{2}\sqrt{1 + \frac{4}{n^2}}$$

When n is large enough, it is possible to write as a first approximation:

(533-8) $$\left[1 - \frac{1}{n}\right]\frac{\gamma_c}{n} < T < \left[1 + \frac{1}{n}\right]\gamma_c$$ [2]

It can be seen that for sufficiently small values of $i - \rho$, the coefficient T can vary only within determined limits once the value of γ_c is given.

---

(1) Condition (220-13).
(2) From (533-3), (533-6) and (533-7).

Values of the Various Magnitudes for Given Values of γ_C and $i-\rho$

534. It is interesting to examine how the various parameters vary with changes in ω and n when γ_C and $i-\rho$ are given by observation.

For example, suppose that we have

$$\gamma_C = 3.5 \qquad\qquad i-\rho = 4\% .$$

To calculate T and Θ_0 we have the two relations (¹)

(534-1)
$$\frac{1-\omega}{1+T(i-\rho)} + \frac{\omega}{[1+T(i-\rho)]^n} = 1-\gamma_C(i-\rho)$$

(534-2)
$$\Theta_0 = [1-\omega+\omega n]\,T$$

Values of T and Θ_0

535. The two following tables give the values of T and Θ_0 for selected values of ω and n (²).

(¹) Relations (526-4) and (526-11).

(²) When n increases indefinitely (a situation which is clearly of theoretical interest only), three cases are possible:

a) *T does not tend to zero.*

Then we have

(1)
$$1 + T(i-\rho) = \frac{1-\omega}{1-\gamma_C(i-\rho)} \qquad \text{for} \quad n=\infty$$

This relation defines the limit of T and it implies

(2)
$$0 < \omega < \gamma_C(i-\rho) .$$

In this case ω is fairly small.

Values of T for $\gamma_C = 3.5$ and $i-\rho = 4\%$

n \ ω	0	0.05	0.10	0.14	0.25	0.5	0.75	1
1	4.1	4.1	4.1	4.1	4.1	4.1	4.1	4.1
2	4.1	3.9	3.7	3.6	3.2	2.7	2.3	2.0
3	4.1	3.7	3.4	3.2	2.7	2.0	1.6	1.3
5	4.1	3.5	3.0	2.7	2.1	1.3	1.0	0.8
10	4.1	3.1	2.4	2.0	1.3	0.7	0.5	0.4
∞	4.1	2.6	1.2	0	0	0	0	0

Values of Θ_0 for $\gamma_C = 3.5$ and $i-\rho = 4\%$

n \ ω	0	0.05	0.10	0.14	0.25	0.5	0.75	1
1	4.1	4.1	4.1	4.1	4.1	4.1	4.1	4.1
2	4.1	4.1	4.1	4.1	4.0	4.0	4	3.9
3	4.1	4.1	4.1	4.1	4.1	4.0	3.9	3.9
5	4.1	4.2	4.2	4.2	4.2	4.0	3.9	3.8
10	4.1	4.5	4.6	4.6	4.4	4.0	3.9	3.8
∞ (1)	4.1	∞	∞	∞	5.1	4.1	3.9	3.8

b) *T tends to zero with* $\omega \neq \gamma_C (i-\rho)$.

Then we have

$$[1 + T(i-\rho)]^n \longrightarrow \frac{\omega}{\omega - \gamma_C (i-\rho)} \qquad (3)$$

which implies

$$\omega > \gamma_C (i-\rho) . \qquad (4)$$

c) $\omega = \gamma_C (i-\rho)$.

In this case, we find

$$T[1 + T(i-\rho)]^{n-1} \longrightarrow \frac{\gamma_C}{1 - \gamma_C (i-\rho)} \qquad (5)$$

Thus T tends to zero, so that relation (5) is verified asymptotically.

$\gamma_C (i-\rho)$ is thus a critical value, and for $\gamma_C = 3.5$, $i-\rho = 4\%$, we have $\gamma_C (i-\rho) = 0.14$.

(1) For infinite n, and from the preceding note, there are three cases:

a) $0 < \omega < \gamma_C (i-\rho)$.

These results are plotted in charts V and VI.

We see that if we consider only the values of n below 10, the only values which it is realistic to consider, then, although the auxiliary parameter T has a wide range of variation when ω and n change, *the values of the parameter* Θ_0 (which are of economic significance since Θ_0 is the length of the amortisation period for $i-\rho=0$) *vary but little.*

Here, T tends to a limit. Consequently Θ_0 increases indefinitely and we have

(1) $$\Theta_0 \sim \left[\frac{\gamma_C(i-\rho)-\omega}{1-\gamma_C(i-\rho)}\right] \frac{\omega n}{(i-\rho)}$$

b) $\gamma_C(i-\rho) < \omega < 1$.

Here, T tends to zero, and

(2) $$\Theta_0 \backsim \omega\, n\, T\,.$$

Now in this case,

(3) $$e^{nT(i-\rho)} \sim \frac{\omega}{\omega-\gamma_C(i-\rho)}$$

so that

(4) $$\Theta_0 \longrightarrow -\frac{\omega}{i-\rho} L\left[1-\frac{\gamma_C(i-\rho)}{\omega}\right]$$

In this case it is clear that according to the value taken by ω, the corresponding value of Θ_0 can take any value greater than γ_C.

For $\omega=1$, we have

(5) $$\Theta_0 \longrightarrow -\frac{1}{i-\rho} L\left[1-\gamma_C(i-\rho)\right]$$

For small $(i-\rho)$ this limit differs little from γ_C.

c) $\omega=\gamma_C(i-\rho)$.

In this case we have

(6) $$e^{nT(i-\rho)} \sim \frac{\gamma_C\, e^{T(i-\rho)}}{T\left[1-\gamma_C(i-\rho)\right]}$$

and since T tends to zero

(7) $$\Theta_0 \sim -\frac{\omega}{i-\rho} LT$$

so that Θ_0 increases indefinitely.

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CHART V

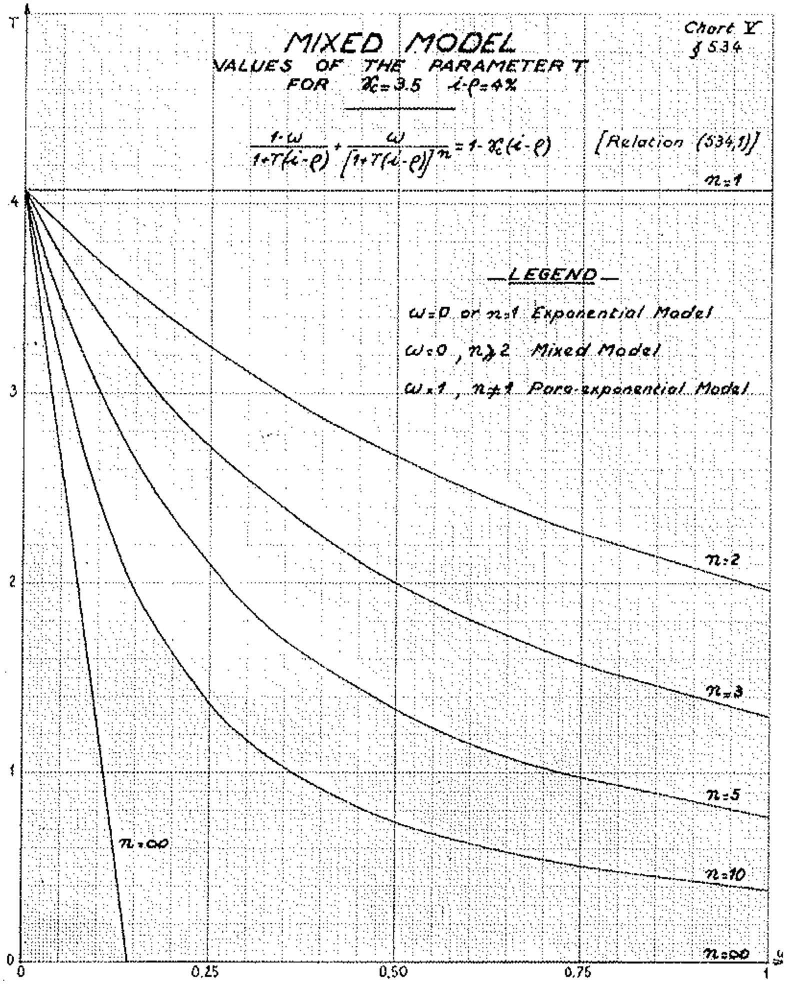
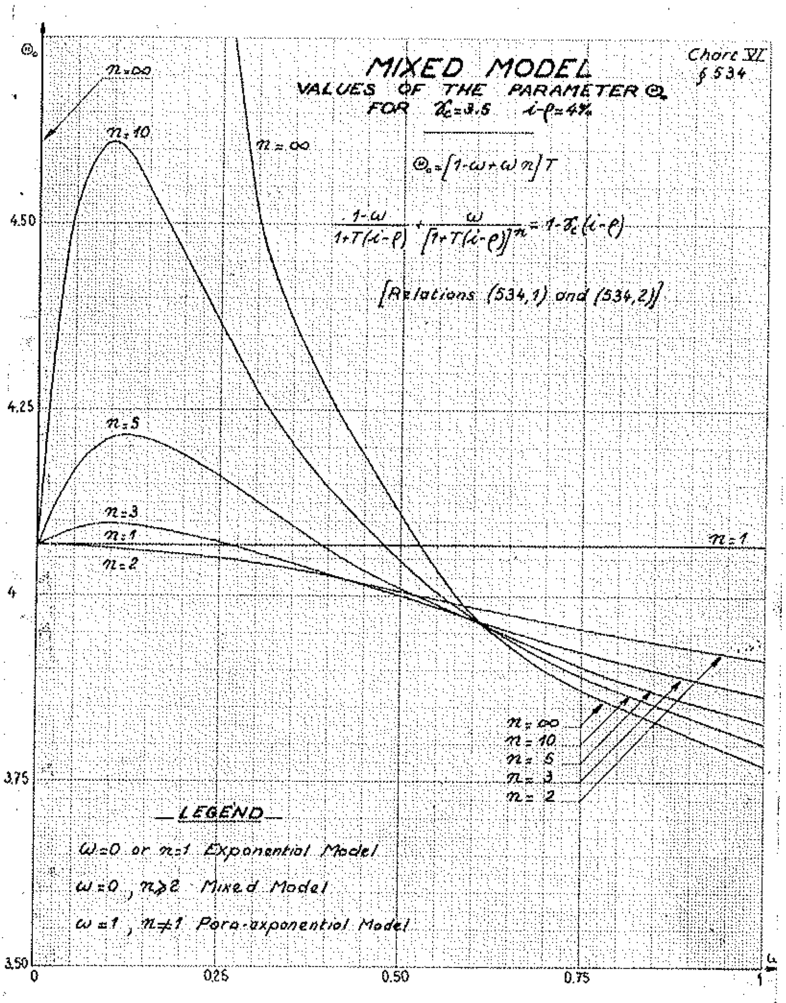


Chart V
§ 534
MIXED MODEL
VALUES OF THE PARAMETER T
FOR $\mathcal{X}=3.5$ $i-\rho=4\%$
$\frac{1-\omega}{1+T(i-\rho)}+\frac{\omega}{[1+T(i-\rho)]^n}=1-\mathcal{X}(i-\rho)$
[Relation (534,1)]
—LEGEND—
$\omega=0$ or $n=1$ Exponential Model
$\omega\neq0$, $n\geq2$ Mixed Model
$\omega=1$, $n\neq1$ Para-exponential Model
n=1
n=2
n=3
n=5
n=10
n=∞
n=∞
T
4
3
2
1
0
0
0,25
0,50
0,75
1

CHART VI



MIXED MODEL
VALUES OF THE PARAMETER Θ₀
FOR τc=3.5 i-ρ=4%
Θ₀=[1-ω+ω n]T
1-ω / 1+T(i-ρ) + ω / [1+T(i-ρ)]^n = 1-τc(i-ρ)
[Relations (534,1) and (534,2)]
Chart VI
§ 534
n=∞
n=10
n=5
n=3
n=1
n=2
n=∞
n=10
n=5
n=3
n=2
LEGEND
ω=0 or n=1 Exponential Model
ω≠0 , n≥2 Mixed Model
ω=1 , n≠1 Para-exponential Model
4.50
4.25
4
3.75
3.50
0
0.25
0.50
0.75
1

Furthermore, we find that $T(i-\rho)$ remains effectively inferior to unity, a necessary condition for the validity of the expansion as a Taylor series of $\psi(i-\rho)$ [1].

Once the values of ω and n become fairly large, $T(i-\rho)$ is relatively small and the expanded series converge rapidly.

Values of Θ

536. From (526-9) and (526-11),

(536-1) $$\Theta = \frac{\dfrac{1-\omega}{1+T(i-\rho)} + n\left[1-\gamma_C(i-\rho) - \dfrac{1-\omega}{1+T(i-\rho)}\right]}{1-u\gamma_C} \; \frac{T}{1+T(i-\rho)}$$

or, again

(536-2) $$\Theta = \left[n - \frac{(n-1)(1-\omega)}{\left[1+T(i-\rho)\right]\left[1-(i-\rho)\gamma_C\right]}\right]\frac{T}{1+T(i-\rho)}$$ [2]

Whence the followiug table for Θ can be constructed [3]

---

[1] § 512, we have $\mu = 1/T$ [relation (510-8)].

[2] The values of T are derived from (534-1) and (534-2).

[3] When n increases indefinitely, three cases are possible in line with the preceding notes

a) $0 < \omega < \gamma_C(i-\rho)$.

In this case T tends to a limit defined by the relation

(1) $$1 + T(i-\rho) = \frac{1-\omega}{1-\gamma_C(i-\rho)}$$

and from this and (536-2) we deduce

(2) $$\Theta = \frac{T}{1+T(i-\rho)}$$

Values of Θ for $\gamma_C = 3.5$ and $i-\rho = 4\%$

n \ ω	0	0.05	0.10	0.14	0.25	0.5	0.75	1
1	3.5	3.5	3.5	3.5	3.5	3.5	3.5	3.5
2	3.5	3.50	3.50	3.51	3.52	3.56	3.59	3.63
3	3.5	3.48	3.48	3.48	3.50	3.56	3.62	3.68
5	3.5	3.42	3.39	3.39	3.42	3.54	3.64	3.72
10	3.5	3.22	3.13	3.13	3.25	3.51	3.66	3.74
∞	3.5	2.37	1.11	0	2.63	3.44	3.66	3.77

i.e.

(3) $$\Theta = \frac{\gamma_c (i-\rho) - \omega}{(1-\omega)(i-\rho)}$$

$$= \frac{\gamma_c}{1-\omega} - \frac{\omega}{(1-\omega)(i-\rho)}$$

whence

(4) $$0 < \Theta < \frac{\gamma_c}{1-\gamma_c (i-\rho)}$$

b) $\gamma_c(i-\rho) < \omega < 1$.

T tends to zero and from (536-1), we have

(5) $$\Theta \sim \frac{\omega - \gamma_c (i-\rho)}{1-\gamma_c (i-\rho)} nT$$

so that

(6) $$\Theta \longrightarrow -\frac{1}{(i-\rho)}\left[1 - \frac{1-\omega}{1-\gamma_c (i-\rho)}\right] L \left[1 - \frac{\gamma_c (i-\rho)}{\omega}\right]$$

Here, Θ can tend to any value according to the value of ω.

For $\omega = 1$ we have

(7) $$\Theta \longrightarrow -\frac{1}{(i-\rho)} L \left[1 - \gamma_c (i-\rho)\right]$$

This is the same limit as for Θ_o.

c) $\omega = \gamma_c (i-\rho)$.

Here T again tends to zero and relation (536-2) gives

(8) $$\Theta \backsim T .$$

Thus Θ_o tends to zero.

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CHART VII

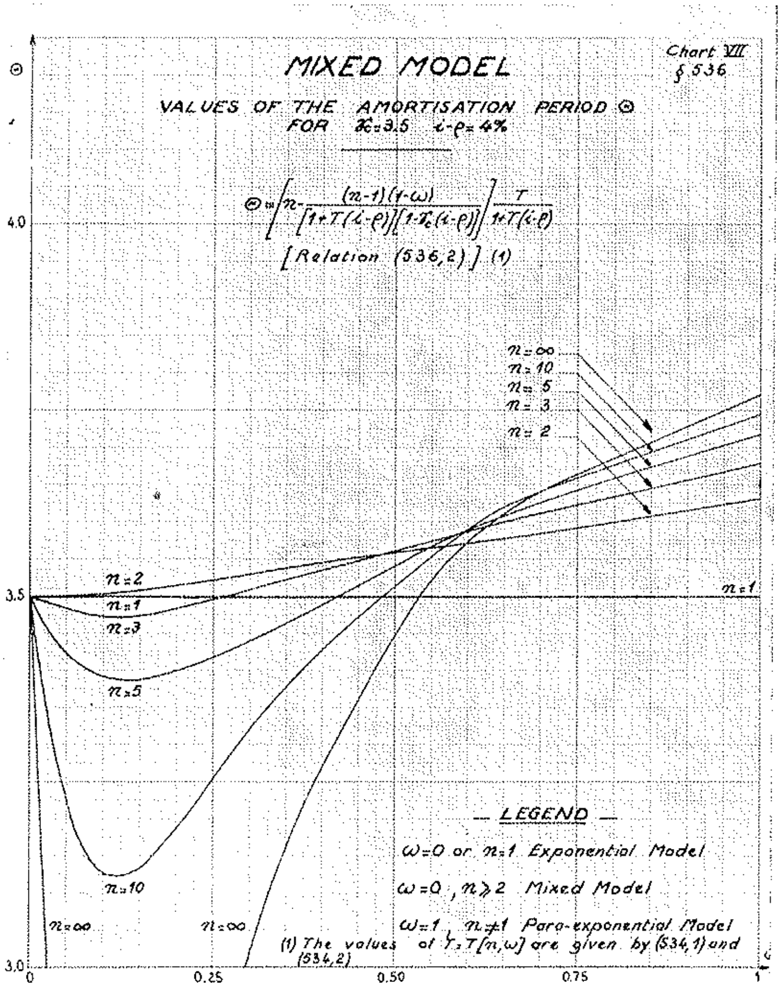
Chart VII
§ 536
MIXED MODEL
VALUES OF THE AMORTISATION PERIOD Θ
FOR $\mathcal{K}_c$=3.5 i-ρ=4%
$\Theta = \left[n - \frac{(n-1)(1-\omega)}{[1+T(i-\rho)][1-T_c(i-\rho)]}\right] \frac{T}{1+T(i-\rho)}$
[Relation (536,2)] (1)
n=∞
n=10
n=5
n=3
n=2
n=1
n=∞
— LEGEND —
ω=0 or n=1 Exponential Model
ω=0 , n≥2 Mixed Model
ω=1 , n≠1 Para-exponential Model
(1) The values of T, T[n,ω] are given by (534,1) and (534,2)
Θ
4.0
3.5
3.0
0
0.25
0.50
0.75
1
ω

These results are plotted in Chart VII. It will be noted that for $n \leqslant 10$, the value of Θ is between 3.10 and 3.80 *whatever the value of* ω. Thus in all cases which are of practical interest, Θ does not vary greatly from its value in the exponential model, i.e. 3.5.

Values of $(\bar{R}_{CM} - \bar{R}_C)/\bar{R}_{CM}$

537. From relation (410-3) and (526-16) we have (1)

$$(537\text{-}1) \qquad g = 1 - \frac{\bar{R}_C}{\bar{R}_{CM}} = 1 - \frac{1}{1-\gamma_C(i-\rho)} e^{-\Theta_0 (i-\rho)}$$

Then, for given values of γ_C and $(i-\rho)$ and the values of Θ_0 calculated in § 535, we can derive the following table.

Values of $(\bar{R}_{CM}-\bar{R}_C)/\bar{R}_{CM}$ *in percent* for $\gamma_C=3.5$, $i-\rho=4\%$

n \ ω	0	0.05	0.10	0.14	0.25	0.5	0.75	1
1	1.2	1.2	1.2	1.2	1.2	1.2	1.2	1.2
2	1.2	1.2	1.2	1.2	1.1	0.9	0.7	0.6
3	1.2	1.3	1.3	1.3	1.2	0.9	0.6	0.4
5	1.2	1.7	1.7	1.8	1.5	1.0	0.5	0.2
10	1.2	3.3	3.3	3.1	2.3	1.1	0.5	0.1
∞	1.2	100	100	100	5.3	1.3	0.4	0.

These results are plotted in Chart VIII. It will be noted that in all cases which are of practical interest, the loss is less

(1) See § 410 above.

CHART VIII

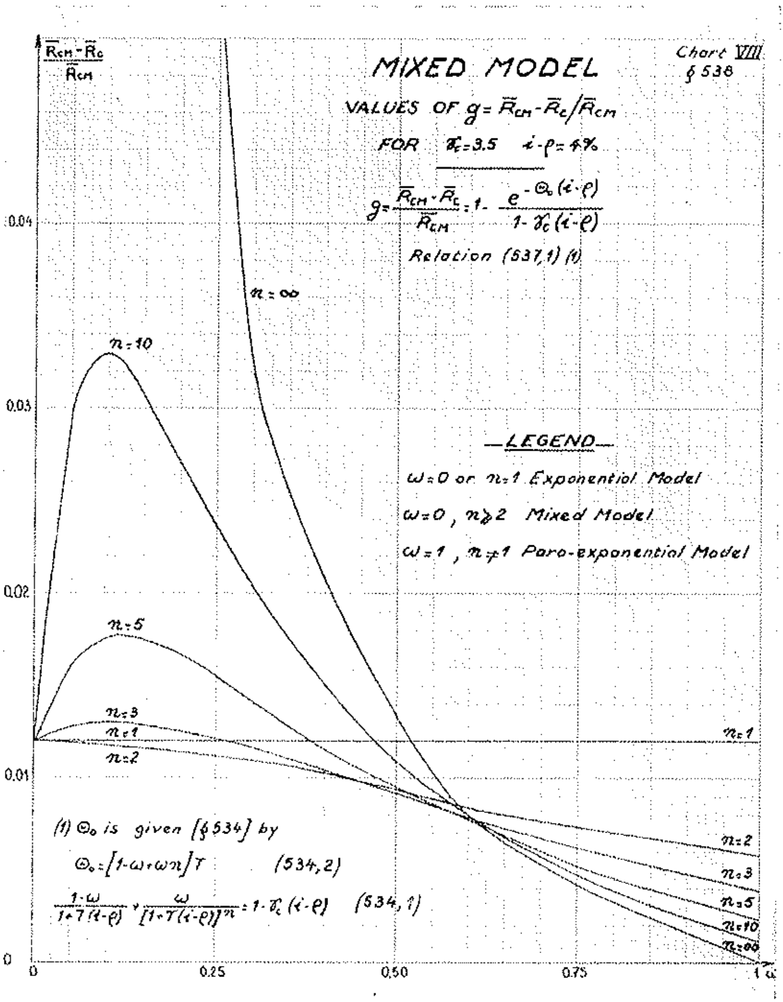

Chart VIII
§ 538
MIXED MODEL
VALUES OF g = R̄cm - R̄c / R̄cm
FOR 𝒯c = 3.5 i - ρ = 4%
g = (R̄cm - R̄c) / R̄cm = 1 - e^(-Θ0 (i-ρ)) / (1 - 𝒯c (i-ρ))
Relation (537,1) (1)
—LEGEND—
ω = 0 or n = 1 Exponential Model
ω = 0 , n ≥ 2 Mixed Model
ω = 1 , n ≠ 1 Para-exponential Model
(1) Θ0 is given [§ 534] by
Θ0 = [1 - ω + ωn] T (534,2)
(1 - ω)/(1 + T(i - ρ)) + ω/[1 + T(i - ρ)]^n = 1 - 𝒯c (i - ρ) (534,1)
n = ∞
n = 10
n = 5
n = 3
n = 1
n = 2
0.04
0.03
0.02
0.01
0
0.25
0.50
0.75
1 ω

than that given by the combination $n = 1$, $\omega = 0$ which corresponds to the exponential model.

Use of the Taylor Series Expansions

538. It is interesting to confront the values which have just been calculated, and those which would be derived using the limited expansions obtained in the main body of the text [1]

$$(538\text{-}1) \qquad \Upsilon_c \sim \Theta_0 \left[1 - \Delta\, \Theta_0 \,(i - \rho)\right]$$

$$(538\text{-}2) \qquad 1 - \frac{\overline{R}_c}{\overline{\overline{R}}_{CM}} \sim \Theta_0^2 \left(\Delta - \frac{1}{2}\right)(i-\rho)^2$$

taking the exact value

$$(538\text{-}3) \qquad \Delta = \frac{1 - \omega + \omega \,\dfrac{n(n+1)}{2}}{\left(1 - \omega + n\,\omega\right)^2}$$

From (538-1) we deduce

$$(538\text{-}4) \qquad \Theta_0 (i-\rho) \sim \frac{1 - \sqrt{1 - 4\Delta\, \Upsilon_c (i-\rho)}}{2\Delta}$$

(1) Relation (240-10) and (240-13).

whence

$$(538\text{-}5) \qquad \frac{\overline{R}_{CM}-\overline{R}_{C}}{\overline{R}_{CM}} \sim \frac{\Delta-\frac{1}{2}}{2\Delta^{2}}\left[1-2\Delta\gamma_{C}(i-\rho)-\sqrt{1-4\Delta\gamma_{C}(i-\rho)}\right]$$

It is interesting to note that for

$$\gamma_C=3.5, \quad i-\rho=4\%, \quad n=5$$

the value obtained from this approximative formulation does not differ from the exact value by more than 18%.

Study of a Particular Case

539. It is interesting to examine the variations of the essential functions in the particular case in which

$$(539\text{-}1) \qquad n=2 \qquad \omega=1/3$$

with

$$(539\text{-}2) \qquad \gamma_C=3.5 \quad \text{for} \quad u=i-\rho=4\% .$$

Then, from the results of § 526, we have

$$T = \frac{3\Theta_0}{4} \tag{539-3}$$

$$\Delta = \frac{15}{16} \tag{539-4}$$

$$\beta(\theta) = \frac{8}{9}\left(1 + \frac{2}{3}\,\frac{\theta}{\Theta_0}\right)\frac{e^{-\frac{4\theta}{3\Theta_0}}}{\Theta_0} \tag{539-5}$$

$$\varphi(\theta) = \frac{\beta(\theta)}{1 - u\gamma_c}\,e^{-u\theta} \tag{539-6}$$

$$\Theta = \frac{1 + \frac{3}{8}\Theta_0 u}{\left(1 + \frac{1}{2}\Theta_0 u\right)\left(1 + \frac{3}{4}\Theta_0 u\right)}\,\Theta_0 \tag{539-7}$$

$$\frac{R_c}{R_\omega} = \frac{1}{1 - u\gamma_c} \tag{539-8}$$

$$\gamma_c = \frac{1}{u}\left[1 - \frac{2}{3\left(1 + \frac{3\Theta_0}{4}u\right)} - \frac{1}{3\left(1 + \frac{3\Theta_0}{4}u\right)^2}\right] \tag{539-9}$$

$$1 - \frac{\overline{R}_c}{\overline{R}_{cM}} = 1 - \frac{e^{-\Theta_0 u}}{1 - \gamma_c u} \tag{539-10}$$

T is determined by (534-1).

[11] *Allais* - pag. 272

The following table summarises the values of the principal quantities for the exponential, mixed and para-exponential models [1] for

$$u = 4\% \qquad \gamma_c = 3.5$$

$\gamma_c = 3.5$ $u = 4\%$	Model		
Parameter	I Exponential	II Mixed	III Para-exponential
	$n = 1$ $\omega = 0$	$n = 2$ $\omega = 1/3$	$n = 2$ $\omega = 1$
T	4.07	3.02	1.96
Θ_0	4.07	4.03	3.92
Δ	1	0.94	0.75
Θ	3.5	3.53	3.63
R_C/R_ω	1.16	1.16	1.16
$\dfrac{\Theta - \gamma_C}{\gamma_C}$	0	0.009	0.04
$\int_0^1 \varphi(\theta)\, d\theta$	0.25	0.23	0.11
$\int_1^6 \varphi(\theta)\, d\theta$	0.51	0.52	0.65
$\int_6^\infty \varphi(\theta)\, d\theta$	0.24	0.25	0.24
$\dfrac{\overline{R}_{CM} - R_C}{\overline{R}_{CM}}$	1.2%	1%	0.6%

(1) We have

$$\int_{\theta_1}^{\theta_2} (A + B\theta)\, e^{-c\theta}\, d\theta = \frac{A}{C}\left[e^{-c\theta_1} - e^{-c\theta_2}\right] + \frac{B}{C^2}\left[(1+C\theta_1)e^{-c\theta_1} - (1+C\theta_2)e^{-c\theta_2}\right]$$

In addition, three charts IX, X and XI are presented showing the functions $\beta(\theta)$, $\varphi(\theta)$ and $\int_0^\infty \varphi(\theta)\,d\theta$ in the three cases. The expressions for these functions are the following (1).

Functions	Model		
	I Exponential	II Mixed	III Para-exponential
	$\omega = 0$	$\omega = 1/3$	$\omega = 1$
$\beta(\theta)$	$\frac{1}{\Theta_0} e^{-\frac{\theta}{\Theta_0}}$	$\frac{8}{9}\left(1+\frac{2}{3}\frac{\theta}{\Theta_0}\right)\frac{e^{-\frac{4\theta}{3\Theta_0}}}{\Theta_0}$	$4\frac{\theta}{\Theta_0^2} e^{-2\frac{\theta}{\Theta_0}}$
$\varphi(\theta)$	$\frac{1}{\Theta} e^{-\frac{\theta}{\Theta}}$	$(A+B\theta)\,e^{-c\theta}$	$4\frac{\theta}{\Theta^2} e^{-2\frac{\theta}{\Theta_0}}$
$\int_0^\infty \varphi(\theta)\,d\theta$	$e^{-\frac{\theta}{\Theta}}$	$\left[A+\frac{B}{C}+B\theta\right]\frac{e^{-c\theta}}{C}$	$\left[1+\frac{2\theta}{\Theta}\right]e^{-2\frac{\theta}{\Theta}}$

with

$$(539\text{-}11)\qquad \begin{cases} A = \dfrac{8}{9\Theta_0(1-u\Upsilon_c)} \\[2ex] B = \dfrac{16}{27\Theta_0^2(1-u\Upsilon_c)} \\[2ex] C = \dfrac{4}{3\Theta_0}\left[1+\dfrac{3\Theta_0 u}{4}\right] \end{cases}$$

(1) Relations (251-2) for $k=1$, (251-3), (539-5), (539-6), (521-4) and (521-6).

CHART IX

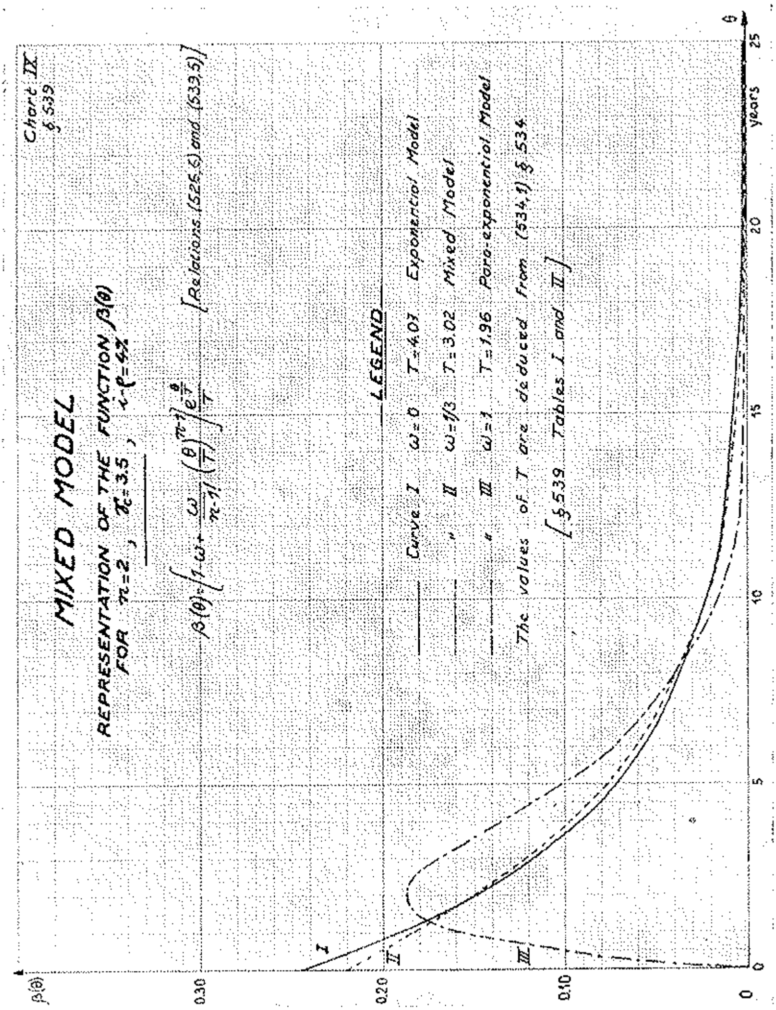
Chart IX
§ 539
MIXED MODEL
REPRESENTATION OF THE FUNCTION β(θ)
FOR n=2 , τc=3.5 , i−ρ=4%
β(θ) = [1−ω+ ω/(n−1)! (θ/T)^(n−1)] e^(−θ/T)/T
[Relations (525,6) and (533,5)]
LEGEND
Curve I ω=0 T=4.07 Exponential Model
" II ω=1/3 T=3.02 Mixed Model
" III ω=1 T=1.96 Para-exponential Model
The values of T are deduced from (534,1) § 534
[§ 539 Tables I and II]
I
II
III
β(θ)
0.30
0.20
0.10
0
5
10
15
20
25
years
θ

CHART X

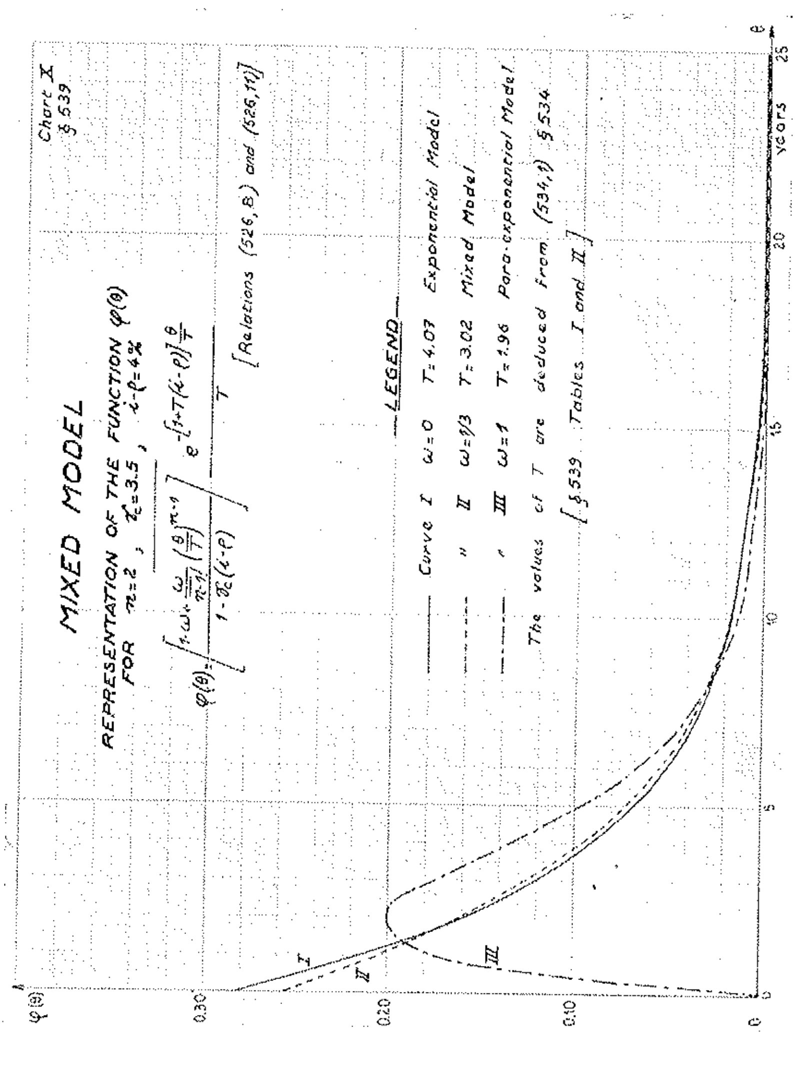
Chart X
§ 539
MIXED MODEL
REPRESENTATION OF THE FUNCTION φ(θ)
FOR n=2, τc=3.5, i-ρ=4%
[Relations (526,8) and (526,11)]
LEGEND
Curve I ω=0 T=4.07 Exponential Model
" II ω=1/3 T=3.02 Mixed Model
" III ω=1 T=1.96 Para-exponential Model
The values of T are deduced from (534,1) § 534
[§ 539 Tables I and II]
φ(θ)
0.30
0.20
0.10
0
5
10
15
20
25
years
θ
I
II
III

CHART XI

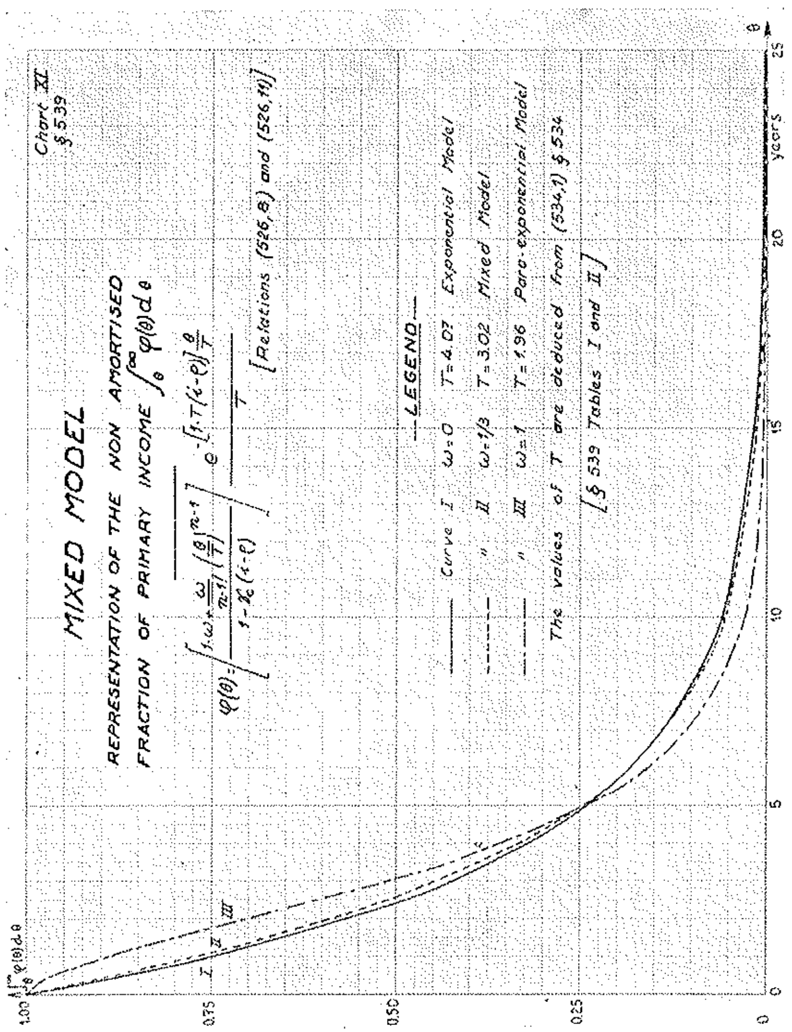
Chart XI
§ 539
MIXED MODEL
REPRESENTATION OF THE NON AMORTISED
FRACTION OF PRIMARY INCOME
[Relations (526,8) and (526,11)]
LEGEND
Curve I ω=0 T=4.07 Exponential Model
" II ω=1/3 T=3.02 Mixed Model
" III ω=1 T=1.96 Para-exponential Model
The values of T are deduced from (534,1) § 534
[§ 539 Tables I and II]
I
II
III
1.00
0.75
0.50
0.25
0
0
5
10
15
20
25
years
θ

It is *really remarkable* that the II curves, which relate to the mixed model, stay very near to the I curves of the exponential model, even though the weight attributed to the exponential element is but 2/3. This is due to the fact that the coefficient Δ in the mixed model has a value very near that which it takes in the exponential model, so that the $\beta(\theta)$ functions of model I and II have approximately the same three first moments (1).

General Comments

540. It is certainly very surprising at first sight to find that for small values of ω and for large values of n, the model considered has very different properties from those of the exponential and para-exponential models, though for $\omega = 0$ it is identical to the exponential model and for $\omega = 1$ identical to the para-exponential model.

For the first and second, for example,

$$\frac{1}{2} < \Delta \leqslant 1$$

$$\Theta_0 \sim \Upsilon_c \qquad \text{for} \qquad i - \rho = 4\%$$

whereas for $\omega = 0.10$ and n sufficiently great, Δ has a value very near 5, and Θ_0 can even become greater than any value (2).

It may first be noted that no illusions should be had about small values of ω, since large values of n can have a greater influence, with correspondingly greater deviation of the model from the exponential.

(1) § 336.
(2) § 532 and 535.

Further, while $\beta(\theta)$ and $\psi(u)$ appear as weighted averages of the corresponding expressions for the exponential and para-exponential models, this is not the case for $\varphi(\theta)$ and the other parameters (1).

Finally, and most important, even if $\beta(\theta)$ and $\psi(u)$, both formally and effectively appear as weighted averages of their expressions

$$\frac{1}{T} e^{-\frac{\theta}{T}} \quad , \quad \frac{1}{\overline{n-1}!} \left(\frac{\theta}{T}\right)^{n-1} \frac{e^{-\frac{\theta}{T}}}{T}$$

$$\frac{1}{1+Tu} \quad , \quad \frac{1}{(1+Tu)^n}$$

for the exponential and para-exponential models (2), the values of the parameter T, and therefore of the corresponding functions β and ψ are *certainly not the same,* since in each of these cases, T is determined by the equation

(540-1) $$\frac{1-\omega}{1+T(i-\rho)} + \frac{\omega}{[1+T(i-\rho)]^n} = 1-\gamma_c(i-\rho)$$ (3)

(1) Relations (510-6), (511-1) and (511-6).
(2) Relations (526-6) and (526-7).
(3) Relation (534-1).

which expresses the fact that theoretical value of γ_C is equal to its observed value, which is the same in all three cases [1], and it is this which generates the large differences noted.

Nevertheless, very large values of n are of no more than speculative interest, and from the foregoing discussion, it may be considered that for $n \leqslant 7$, the properties of the model studied are extremely close to those of the exponential model, *whatever the value of* ω.

It follows from this that if we can consider only the first few terms of the expansion of $\beta(0)e^{\mu 0}$ *as a Taylor series, the general properties of the model will be very close to those of the exponential model.*

[1] For the exponential model, we must put $\omega = 0$ in relation (540-1); for the para-exponential model, $\omega = 1$, and for the mixed model $\omega = \omega_1$.

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(¹) The second edition is identical to the first, differing only by the presence of an introduction noting the new contributions which were made in the first edition.

Document 61, 70 pages. Because of copyright, only the theoretical part has been published in the Bulletin of the I.S.I., Vol. XXXVIII, 2, pp. 3-27.

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(1) This study contains a large bibliography, presented in a systematic fashion, which the interested reader may usefully consult.

(2) It is necessary to mention an unfortunate printers error which got into the text of my Econometrica paper *after* correction of proofs.

The following correction was printed in a subsequent number of « Econometrica » (vol. 31, n. 4, Oct. 1963, p. 784).

Instead of « ... the capitalistic process P (shown on the left on the next page) tends to the asymptotic process Pa (shown on the right) with... » read: « ... the capitalistic process P tends to the asymptotic process Pa with... ».

DISCUSSION

FISHER

Professor ALLAIS' interesting paper makes essentially one empirical test. The model predicts two items to be fairly constant. One of these is γ, the capital output ratio, which the theory predicts should vary rather weakly, and the other is θ_0 which the theory predicts to be a constant. Indeed γ is supposed to vary weakly because it is essentially a function of θ_0 and some things that don't move very much. Professor ALLAIS secures numbers on these two magnitudes.

Now, when a theory predicts something to be constant it is hard to know how to evaluate the test that is used. One has to be able to say how constant is constant and there is no standard for this. The only possible comparison that can be made, I should think, is a relative one. Professor ALLAIS' theory predicts that θ_0 should be constant and γ should vary slightly; in fact, θ_0 varies substantially more than does γ. θ_0 moves by something of the order of 17.5 per cent as compared with 5.8 per cent for the figures for γ for the United States. This takes the ratio of the range to the mid point of the range as the basis for the calculations which are imprecise. If the theory were right, it should be the other way around.

Now this is rather serious since it is almost the only empirical test performed by ALLAIS. I can well understand that one cannot rest much on an argument such as this one, since these numbers are very rough and are subject to substantial errors of measurement. Indeed, as ALLAIS has shown, even plausible errors of measurement can lead to this result. On the other hand, I am unable to understand why these numbers are appropriate for ALLAIS to use and not appropriate for me to criticize. Professor ALLAIS insists upon saying that everything goes according to his model. In fact, this does not go according to his model, and while that may be the result of measurement error, it may equally not be. It is, of course, all too common for an investigator to claim that numbers which apparently point opposite to his theory are subject to measurement error. Professor ALLAIS, however, is managing to do more than that. First he puts forth these numbers as supporting his theory; then, when I point out that in fact they do not support it, he claims that is due to measurement error; finally, he claims that in fact they do support it. Professor ALLAIS is working both sides of the street here and I am afraid I cannot understand his somewhat lengthy argument on this point.

ALLAIS

First, in my opinion it is impossible to attach any importance to these slight differences because the precision of the different estimates is quite small and it is difficult to derive any conclusion from small deviations which are probably of a random character.

For instance, if you consider the 58 COLIN CLARK figures for 21 countries cited by me in Table V, p. 54 of my Tokyo paper (ALLAIS, 1960 A), the range for γ is quite large, from 0.83 to 8.48. The median is 3.54 and the coefficient of variation about it is 22%.

If you consider each of the 58 figures, this is certainly wrong and only the median of the lognormal distribution has any real meaning.

[11] *Allais* - pag. 284

2. My second point is that the difference you underlined is not as great as you say.

I gave the detailed figures in my « Econometrica » paper of October 1962, pages 714 and 715. They are as follows:

TABLE I

United States			
Years	γ	i	Θ_0
1880	3.13	6.47	3.82
1890	3.87	5.68	4.79
1900	3.69	5.18	4.42
1906	3.37	5.56	4.03
1910	3.38	5.81	4.08
1913	3.44	6.06	4.21
1923	3.34	6.57	4.16
1929	3.46	6.48	4.33
1937	3.88	5.15	4.70
1950	3.40	3.86	3.65
1955	3.30	4.25	3.59
1956	3.35	4.57	3.69
Average	3.46	5.47	4.12
Average relative deviation (1)	5 %	13 %	8 %

(1) For a quantity y_i the average relative deviation is $\overline{(y_i - \bar{y})}/\bar{y}$ where $\bar{y}$ is the average of the y_i.

TABLE II

United States, Great Britain, France - 1913			
	γ	i	Θ_0
United States	3.44	6.06	4.60
France	3.83	4.84	4.21
Great Britain	3.67	5.00	4.50
Average	3.64	5.30	4.44
Average relative deviation	4 %	12 %	3 %

Thus, the average relative deviation is respectively 5% and 8% for γ and Θ_0 and for the United States only and 4% and 3% for the United States, Great Britain and France in 1913. In fact, in the first case, the range of Θ_0 is slightly greater than the range of γ for the first table. This order is reversed for the figures of the second table.

I repeat that in my opinion we cannot attach any importance to the decimals. Perhaps some years from now it will be possible to have exact figures for the decimals but my conviction is that this is impossible at the moment. And so far as the figures for the United States are concerned, I must stress that there is some bias associated with the estimate of the rate of interest. It is very difficult indeed to estimate the pure rate of interest i. For example, after the war, American monetary policy was such that the rate of interest was maintained at artificially low levels and in this way the rate of interest did not reach the value which it would have had it there had been free play of the market mechanism. This explains the slight tendency mentioned by FISHER.

3. But that is not the real point. The real point is that the results which are found show a striking agreement as far as orders of magnitude are concerned; and at all events these results must be explained. The proposed theory can predict small variations of Θ_0 and γ and in fact we verify that the estimates do not vary very much.

4. My fourth point is the following. I am completely aware that it is impossible for me to say that this model is the only one which can explain the facts, because the empirical data we have are insufficient for any final and definite conclusion to be derived. I say only that the course of events is what it would be if this theory were correct, and no more than this. This theory and the model which illustrates it are compatible with all the known facts.

I don't say that this theory will continue to hold. Nobody knows this. I simply suggest that this theory can explain the features that we find in the empirical data in a simple way. This statement is valid for any theory whatever. It is impossible to declare for any theory « that is the right and definitive theory ».

We do not know if the Newton Theory of gravitation or the Einstein Theory of gravitation is correct or not — we don't know that — what we can say is that the course of events is consistent with their correctness, at least as first approximations, and I don't say anything more.

I would add that this theory and its associated may not be right, but at least, as far as I have been able to judge, they have the merit of having been able to force people to think about many interesting features of reality (see my paper § 337).

WOLD

I am extremely impressed by the whole of Professor ALLAIS' presentation. I have received many books and papers by ALLAIS, but I have never had the opportunity to listen to an exposition like this. Maybe this is the first time that Professor ALLAIS gives such an integrated exposition, and if so I am very happy to be one of the first listeners. It adds to my admiration that his material, from the criteria of the Study Week, is very important. It is a question of course to what degree his views will be accepted, but there is no doubt that this is the type of material which we are supposed to present and discuss.

My own comments and remarks are very small. My first question mark in the margin refers to your notations for the capital. If the total is divided in parts, then the usual notation is that the parts have indexes, and the total has no index. In your system I find that the total has one index, and one of the parts has no index, and this is a little confusing.

ALLAIS

It was, I think, JEVONS who used this characteristic curve for the first time, but for the case of a stationary equilibrium and in triangular form. At the end of the last century, BOEHM-BAWERK again used such a curve but in rectangular form. Subsequently a German, STACKELBERG, and a Frenchman, BOUSQUET, again used the concept of the characteristic curve for stationary process with rectangular and triangular form. If they did not go farther, the explanation is, I think, that they were unable to treat this quite difficult question mathematically. I have given tbe references to their works in the bibliography of my « Econometrica » paper with some comment.

The concept of the characteristic curve is thus quite classic. What I have done is first to calculate the different macroeconomic quantities R, C, etc. and secondly to generalize this concept for the dynamic case.

What is completely new is the calculation of the real consumed national income, the principles of which I gave in my book « Economie et Intérêt » in 1947. I stress that the hypotheses underlying this calculation are quite natural and not very strong (see § 119 for the general theory and § 221 for the general model).

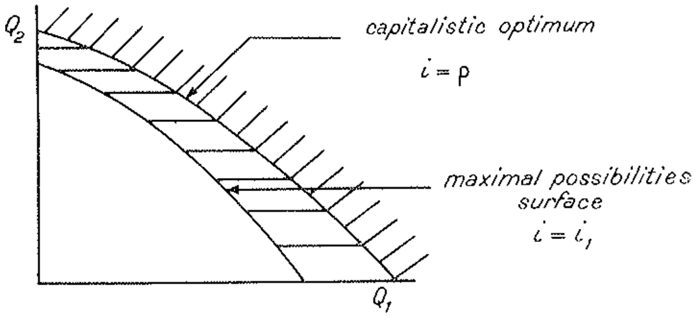
I can underline one point more. In the present paper, because the value of reproducible capital appeared so often, it seemed better to me to use the letter C without a subscript to represent reproducible capital and to use C_T to represent the total value of reproducible capital and land. (On the contrary, in my book « Economie et Intérêt », I have followed the ordinary way).

WOLD

Thank you for that explanation. Then I wonder whether it is correct to understand from your analysis that thanks to the free working of the forces in the capitalistic system, we are rather near the optimum of maximal possibilities?

ALLAIS

A complete explanation would be too long. My point is only that if we assume that the total real value of capital is at a certain level so the surface of maximal possibilities has a certain position, as indicated on the following chart



Q_1 and Q_2 are the productions of two consumption goods, for instance, butter, shoes, anything. If capital is varying, this frontier is varying also and my point is that a maximum frontier does exist, and that it characterizes an optimum capitalistic structure. This question was studied quite extensively in my 1963 Cambridge paper.

In fact, the conclusion can effectively be that we are very near the optimal situation as far as capital is concerned; that is, so far as the capital distribution over time is concerned (see my paper § 410-411). But so far as other factors are concerned, this is not so and specifically, the difference between French and American productivity can be explained only if the other factors are taken into account (see my paper § 430-433).

LEONTIEF

In applying his analytical approach to interpretation of differences and similarities in the relative magnitnde of the labor and capital inputs in several countries, Professor ALLAIS apparently assumes direct comparability of the units in terms of which the labor inputs are measured in different areas. Does this imply that a man year of labor in the United States translated into relevant efficiency units is equivalent to a man year of labor in Canada, in France or in India? If it does not, how would a correction for differences in labor efficiency affect his comparisons?

Allais

I don't suppose anything. The only relevant point is the fact that there exist great productivity differencies. To explain them, different factors must be examined: labor, natural resources and capital. The present paper is limited to the study of the possible influence of capital. In my book « L'Europe Unie », published by Calmann Levy in 1960, I have presented for the United States and France a general analysis taking into account the different factors, the discussion of which is not possible in the limited framework of this discussion.

Nevertheless as far as I can judge, my personal position differs slightly from yours. I think you agree that capital is not playing as great a role as many people believe and I am happy that you have arrived at the same conclusion as I. But so far as labour is concerned, it seems to me that may be we should be quite cautious because there are two things to consider. The first one is the level of the productivity of persons and the second is the efficiency of the economic system. My own view on this questions, so far as the United States and France are concerned, is that it is impossible to conclude that American workers are superior to French. Perhaps the best example I can give is that when I deliver a lecture in France, people say « Oh, French workers are superior to the American workers without any doubt » and in the United States they say exactly the contrary. That American workers are better. In my opinion there is no difference at all. What is different is the economic system, the pressure of competition, the price mechanism and so on. And my conclusion is that the main reason for the difference in productivity between the United States and France, so far as I can judge, is the play of the economic system.

In any case I want also to answer some questions you raised and you did not mention just now.

First I cannot affirm that there is only one explanation why the capital output ratio is practically constant. In my book « Economie et Intérêt » (Chapter IX), I presented some other arguments relating to the CASSEL theory and taking into account the length of life, and I think these arguments are quite appealing. I would not say that there is only one explanation. I would suggest only that the stability, for practical purposes, of the capital output ratio can be explained by the constancy of Θ_o which represents the intellectual difficulty of conceiving roundabout production processes. My present theory is thus at least compatible with the facts.

Second, the coefficient Θ_o is, as I have already said, an index of the intellectual difficulty of conceiving roundabout production processes and psychologically, the thought that the coefficient Θ_o is constant over time because over time people have the same intellectual capacity to conceive roundabout processes, is appealing to me. Of course, I am suggesting an explanation, not offering a proof.

Thirdly, LEONTIEF spoke about a possible cumulative effect. I would say that I have done some quite interesting statistical research on the question of the intellectual capacity of people over time. More specifically, I have tried to assess the rate of growth or scientific and technological progress over time since the 12th century. I have considered the number of major scientific and technological discoveries per century and the result of this statistical analysis, which I cannot develop here, is that we can conclude that people's intellectual capacity has remained exactly the same since the 12th century. But, if one consider real wages, there have been tremendous fluctuations. Thus the commonly held view: that the extraordinary development of the West in the last century is due to new inventions is very questionable indeed. My opinion is that in fact the economic system has played a very great role.

Thus, if I could briefly answer to the LEONTIEF point, I would say that there is no reason to believe that people are more intelligent now than they have been yesterday.

In addition, as far as a cumulative effect is concerned I have shown in my 1963 Cambridge paper that, subject to very general and appealing conditions, it is impossible to expect an indefinite increase of real national income to result from an indefinite increase of real capital.

THEIL

Regarding the problem of the accuracy of statistical data, there is a difficulty which econometricians have to face and which is due to the considerable margin of uncertainty that economic statisticians have to face. For example, it may happen that the « best » estimate, given the statistical evidence, which an economic statistician can produce implies a capital-output ratio which is excessively large compared with most published capital-output ratios. We should not be surprised when in such cases the statistician decides to replace his « best » capital by a lower figure and thus contributes to a too homogeneous picture of the stock of published capital-output ratios. In the same way, if the economic statistician knows or feels that he knows that the marginal propensity to consume is about 0.8, and if he finds for a particular year that the consumption change is very far from 80 per cent of the income change, he may decide to adjust his figures. This is in my view one of the reasons why our picture of published correlation coefficients tends to be too optimistic.

ALLAIS

These different observations are very interesting and I think they should be discussed in the general meeting.

First, as far as the estimates of capital are concerned, it is certain that the estimates we have now are very questionable. But 20 or 30 years ago the same was true for national income figures and many people were saying at that time that this sort of calculation was useless and had no meaning at all, and so on.

My second point. What Professor THEIL has said is very interesting indeed, as far as the proposal for some general statement concluding this colloquium is concerned. I have read in the introduction to the Study Week that econometrics has made much progress and so on, and to-day we hear from a very competent colleague that the work of every statistician in the world may after all be very questionable, not only from the scientific point of view but from the point of view of honesty. What you have said is terrible indeed. Terrible. For if you were right, this would mean that statisticians are dishonest. Personally I do not accept this point. I believe that statisticians do not revise their estimates when they differ from other estimates, and I can supply very good proof of it. For example, there are differences between the various estimates COLIN CLARK has given for the capital output ratios of different countries at different times. These estimates are indeed very different from one country to another and from one time to another. As you know, COLIN CLARK has not made any personal calculations on this. The figures published by him are estimates made by different statisticians around the world, using very different methods and taking into account statistical materials which are absolutely not comparable. Not only are these figures not the same, but their order of magnitude is absolutely different. We therefore cannot suspect the statisticians of having modified their estimates in order to be in agreement with the other evaluations. The contrary is the case.

But if these figures are absolutely different, they are distributed lognormally and their median is practically the same as median for the United States from 1880 to 1956. Thus we can conclude that there is an unquestionable reality behind these very different estimates and we can conclude that the differences are due to some random influences.

If I can answer to the FISHER point, I would say that the agreement between the averages for the United States, for the world and for 1913 (§ 321 and 323 of my paper) is in any case really striking. Of course there are some differences but many other factors are operating and what is astonishing is not that some small differences exists, but that these differences are in relative terms so small.

And finally I would add that some of the arguments I have heard are very strange indeed. If the statistical data and my theory were in total disagreement, nobody would have said, « Oh, it is clear: if there is disagreement, the data are wrong ». They would have said, and in my opinion rightly, « The theory is wrong because it does not agree at all with the facts ». So, if there is concordance here between the theory and the facts, we must recognise that there is an agreement, and this is certainly a point in favour of any theory at all times. Perhaps it could be argued that this agreement is due to chance. I am very doubtful about such a proposition. It is quite unbelievable that such coherence between figures for the United States between 1880 and 1956 and for Great Britain or France in 1913 and for 58 values for 21 countries could arise by chance.

THEIL

I am afraid that I did not succeed in making my point sufficiently clear. What I was trying to say is that in a great many cases economic statisticians have to face the problem that the data which they collect imply sizeable uncertainties. The uncertainty may either be due to the inferior quality of the data themselves, or to their imperfect relevance with respect to the variables for which they are used, or to their sample character. When there is a large range of uncertainty it is evidently rather arbitrary which particular point estimate will be published. Under such circumstances it is not at all unreasonable that other considerations are taken into account (besides the uncertain numerical data) when a decision has to be made as to which figure will be published. If these considerations are based on economic analysis we obtain the picture which I tried to sketch.

ALLAIS

I agree with what Prof. THEIL has said. That is my own position. But I am doubtful that there is any real tendency for statistician to adjust their estimates by taking other people's results into account.

(*)

ALLAIS

In presenting my paper as briefly as possible I wish to stress only some very important points on which remarks were formulated in the LEONTIEF Group's discussion (1).

1) The main definitions are given on Table I.

2) The theory I present uses the concept of characteristic curves (Table II) which was first developed by JEVONS. There are two curves, the first for the production process and the second for the amortization process. The first represents the inputs $\hat{r}_\theta d\theta$ of primary income (the services of Labour and land) supplied at time t-θ and emerging in consumed national income R at time t. The second represents the inputs $r_\theta d\theta$ supplied at time t and emerging in consumed income at time $t+\theta$.

In a stationary process the two curves are simmetrical but in

(*) Discussion in plenary session.

(1) During this discussion, large sheets were put on the blackboard. The main definition and equation were presented on these sheets.

The contents of these sheets is indicated in the tables in the abstract of Professor ALLAIS's paper, and the following discussion refers to these tables.

a dynamic process they are different and the analysis is much more complicated.

In the first part of the paper I calculate different macroeconomic quantities, R national income, C reproducible capital, γ and γ_C capital output ratios, as functionals of the two functions

$$\hat{\varphi}(\theta) = \frac{\hat{r}_\theta}{\hat{R}_\omega} \qquad \varphi(\theta) = \frac{r_\theta}{R_\omega}$$

The relations obtained (Table III) are simply accounting identities.

It is assumed that there exists a valid index $\bar{R}_C$ of real consumed national income such that

$$\frac{\delta \bar{R}}{\bar{R}} = k \frac{\sum y \,\delta Y}{\sum y Y}$$

where the Y represent the primary inputs, the y their prices and k the homogeneity coefficient of the production function.

From this hypothesis it is possible to derive the general formula (117-18) of the paper expressing $\delta\bar{R}$ as a function of the $\delta\hat{\varphi}$

$$(117\text{-}18) \qquad \frac{1}{k}\,\frac{\delta \bar{R}_c(t)}{\bar{R}_c(t)} = \frac{\int_0^\infty \delta\hat{\varphi}(t,\theta)\, e^{\int_{t-\theta}^{t} i(u)du}\, d\theta}{\int_0^\infty \hat{\varphi}(t,\theta)\, e^{\int_{t-\theta}^{t} i(u)du}\, d\theta}$$

[11] *Allais* - pag. 296

From this relation it is possible to demonstrate that real consumed national income is maximised when the rate of interest i is equal to the rate of growth ρ of primary income (Table III).

4) It is possible to illustrate the theory by a very general model (Table IV). I assume only that the production function is logarithmically linear and that the elasticities

$$\beta(\theta) = \frac{d\bar{R}_C}{\bar{R}_C} \Big/ \frac{d\hat{r}_\theta}{\hat{r}_\theta}$$

can be considered over very large range as constant overtime (i.e. independent of t but dependent on θ).

Thus we derive the expression for real consumed national income as

$$L\bar{R}_C(t) = L\alpha(t) + \int_0^\infty \beta(\theta)\, L\hat{r}_\theta\, d\theta$$

and we can calculate the different macroeconomic quantities in function of the Laplace transform $\psi(u)$ of the elasticity $\beta(\theta)$. In particular we finally reach the expression

$$\frac{\bar{R}_C}{\bar{R}_{CM}} = \left[\frac{1-\gamma\rho}{1-\gamma i} e^{-(i-\rho)\,\Theta_0}\right]^k$$

where γ is the capital output ratio C/R, i the rate of interest, ρ the rate of growth of primary income R_ω (the Services of Labor and national resources), k the coefficient of homogeneity of the production function and Θ_0 a constant equal to the value of Θ for $i=0$. $\bar{R}_{CM}$ is the maximum value of $\bar{R}_C$ attained for $i-\rho=0$.

For the maximum value $\overline{R}_{CM}$ of the real national income consumed we find

$$\overline{R}_{CM} = \alpha(t) \left[\frac{R_\omega(t)}{e\Theta_0} \right]^k e^{-k\Theta_0 \rho} \tag{251-17}$$

where e is NAPIER's constant, and $\alpha(t)$ and $R_\omega(t)$ are given. This formula enables the influence of ρ on the maximum value R_{CM} to be studied (§ 420-421).

The expression obtained for $\overline{R}_C/\overline{R}_{CM}$ is very interesting since it depends only on two unknown quantities k and Θ_0. But it is easy to show that the order of magnitude of k and Θ_0 are respectively 1 and γ_C which is a statistical datum.

The expression for $\overline{R}_{CM}$ shows that of all the processes of growth ($\rho \geqslant 0$) the most advantageous one is that for which the rate of growth of primary income is zero.

For small values of the rates i and ρ it is possible to expand the different macroeconomic quantities as TAYLOR series as functions of three constants k, Θ_0 and Δ if the first terms of the TAYLOR expansions only are considered (§ 220, 240 and 241).

In particular

$$\frac{\overline{R}_C}{\overline{R}_{CM}} \sim 1 - \frac{k}{2} \Theta_0^2 (i - \rho)^2 \tag{251-18}$$

5) If we assume, as is quite natural, that the elasticity $\beta(\theta)$ is decreasing exponentially i.e.

$$\beta(\theta) = \frac{k}{\Theta_0} e^{-\frac{\theta_0}{\Theta_0}} \tag{250-5}$$

(exponential model, Table V) we find that every macroeconomic quantity can be expressed as a function of $R_\omega(t)$, $\rho(t)$, $i(t)$ and of two constants only: k and Θ_0. In particular we have

$$\text{(251-6) and (251-12)} \qquad \gamma = \frac{C}{R} = \hat{\Theta} = \frac{\Theta_0}{1 + \Theta_0 i}$$

$$\text{(251-15)} \qquad \frac{\overline{R}_C}{\overline{\overline{R}}_{CM}} = \left\{ [1 + \Theta_0 (i - \rho)] e^{-\Theta_0 (i - \rho)} \right\}^k$$

In this case we have

$$\Delta = 1$$

and in the appendix I have shown that under very general hypotheses Δ remains in the neighbourhood of 1. Thus there is reason to believe that, at least as a first approximation, reality can be represented by the exponential model.

If we consider the meaning of relation (250-5), Θ_0 appears as an index of the intellectual difficulty of conceiving roundabout production processes. Thus we can consider it as being practically constant, and this hypothesis seems to be confirmed by the statistical data (§ 321).

6) With the formulae obtained it is possible to estimate (Table VI and VII) the parameter Θ_0 since

$$\text{(321-1)} \qquad \Theta_0 = \frac{\gamma}{1 - i\gamma} \qquad \gamma = \frac{C}{R}$$

We find that the order of magnitude of Θ_0 is 4.

For k there are many reasons for believing that its value does not differ very much from unity.

A precise estimate of Δ appears to be particularly difficult to make, but as I have already said, the analysis in the appendix shows that under very general conditions, we can consider that

$$\Delta \sim 1$$

7) The general model and its exponential variant appear to be justified by the facts both as to their hypotheses and their consequences.

The third part of my paper shows that the course of event to what one would expect if the general model were be correct, and as far as can be judged, reality does not differ very much from the exponential variant of the general model.

I must particularly stress the striking agreement between the conclusions of the theory and the practical constancy of the estimates of the capital output ratios γ and of the estimates of the coefficient Θ_0 (§ 321, 323 and 324) (Table VII).

8) Finally four different applications are given in the fourth part of my paper (Table VIII). These are the following:

First, it is possible to estimate what could be obtained from an increase in the capital output ratio $\gamma_C = C/R_C$. It is shown that developed countries are in the neighbourhood of an optimum capitalistic structure (§ 410-411).

Secondly, the diminution of the real national income as a consequence of population increase is estimated to be of the order of magnitude of $k\Theta_0\rho$. For the United States, this figure is of the order of magnitude of 7% (§ 420-421).

Thirdly, the model is used to study the problem of estimating the influence of the capitalistic structure on differences in productivity between France and the United States. As far as can be judged, existing differences must be explained by other factors (§ 430-433).

Fourthly, it is possible to show the orientation which is required in the development policy of underdeveloped countries.

9) I have finished my own exposition, but there is just one more point, I would like to discuss. This point is related to some of the other papers presented here, namely HAAVELMO's paper, LEONTIEF's and THEIL's. These three papers accepting as a valid hypothesis that if real capital is increased there is a proportionate increase in real national income.

This means that they assume the relation

$$\bar{R} = \gamma \bar{C} \qquad (a)$$

as a production function, where $\bar{R}$, $\bar{C}$ and γ are respectively real national income, real capital and the capital output ratio. This relation is apparently derived from the practical constancy of the capital output ratio

$$\gamma = \frac{C}{R}$$

but the deduction is in fact a very questionable one, *empirically and theoretically.*

First, as I said in the discussion on LEONTIEF's and HAAVELMO's papers, from an empirical point of view this hypothesis is completely unacceptable.

The only data we have, at least to my knowledge, are the results obtained by DOUGLAS and his followers. According to them, the elasticity of real income with respect to real capital is of the order of magnitude of 0.2 to 0.25. If we use these values in the HAAVELMO, LEONTIEF and THEIL models we will find very very different results. In particular LEONTIEF's results are not as optimistic as they appear in his paper.

From a theoretical point of view it is possible to show that the proposition: « real income is proportional to real capital » cannot be derived from the proposition: « the capital output ratio is practically constant ».

I can illustrate this proposition in using my own model which in any event is mathematically consistent. In the exponential case of this model we obtain two results simultaneously

a) the capital output ratio

$$\gamma = \frac{C}{R} = \frac{\Theta_0}{1 + \Theta_0 i} \qquad (323\text{-}1)$$

changes little if we consider the usual range of variation of i;

b) real national income

$$\bar{R} = \bar{R}_M \left[\frac{\Theta_0}{\gamma} e^{1 - \frac{\Theta_0}{\gamma}} \right]^k \qquad (251\text{-}16)$$

has a maximum value $\bar{R}_M$ *whatever the value of real capital* $\bar{C}$.

Thus, it is impossible to admit the derivation of relation (*a*) from the observed constancy of the capital output ratio as a valid and general proposition.

10) Finally, two main objections were raised in the LEONTIEF group's discussion.

The first was that one cannot have great confidence in the statistical data so that the agreement between my theory and the facts should be questioned. This argument is quite singular. In fact, there is a striking agreement between three sorts of data.

For France, Great Britain and the United States in 1913 we find an average value of γ of 3.64. For the U.S.A. from 1880 to 1956, we find 3.46 and the median of 58 values given for 21 countries by COLIN CLARK is 3.54 (§ 323-1). Such agreement cannot be due to chance.

Ordinarily if we have a theory and the facts are in disagreement with this theory, the theory is rejected. But here the objection is quite different. It is that if there is agreement we cannot infer anything because the data are questionable. Perhaps we ought to reject the facts because they are in striking agreement with the model?

The second objection was rather more justified. Professor FISHER noticed that the range of variation of Θ_0, was a little greater than the rate of variation of γ. And, in the light of this theory, we should have smaller variations for Θ_0 because we have

$$\frac{1}{\gamma} = \frac{1}{\Theta_0} + i \tag{323-1}$$

Θ_0 being a constant and γ a function of i.

In fact, I have given the relative deviation of γ and Θ_0 from their averages for the United States in my Econometrica paper (1962). I found 5% and 8%; the difference is not so great. For United States, Great Britain and France in 1913 we have 4% and 3%. Here it is true that the first figure is greater than the second.

But in any case the extent of the range of Θ_0 can be explained by the difficulty of estimating the rate of interest correctly. In fact we have

$$\frac{\Delta\Theta_0}{\Theta_0} = \frac{\Delta\gamma}{\gamma^2} + \Delta i = \frac{\Delta\gamma + \gamma^2 \Delta i}{\gamma^2}$$

Thus, an error Δi has the same effect as $\Delta\gamma/\gamma^2$, that is we must compare $\gamma^2\Delta i$ with $\Delta\gamma$. Thus the error in γ can be 0.1 and the error in i can be 0.015 but

$$\gamma^2\Delta i = 0.15 .$$

It can thus be seen that the influence of an error in i can be great and as you know it is very difficult ot estimate the pure rate of interest.

We con derive the same conclusion in another way. For the United States, we can calculate the theoretical value of i

$$i_T = \frac{1}{\gamma} - \frac{1}{\Theta_0}$$

where Θ_0 is the average value 4.12 found for Θ_0. Thus we find the following table

United States			
Periods	Estimated value i_e in %	theoretical value i_T in %	difference $\Delta i = i_e - i_T$
1880-1900	5.03	3.82	1.21
1906-1913	5.06	5.14	−0.08
1923-1937	5.37	3.82	1.55
1950-1956	2.25	5.58	−3.32

Thus, for the period 1880-1956, the differences Δi are quite small and do not exceed 1.5%. But for 1950-1956, the difference is very great. The reason is, in my opinion, the fact that American monetary policy has maintained the rate of interest on bonds at an artificially low level which did not correspond to the general economic situation. Thus, the rate of interest on investment was certainly much higher than the rate of interest on bonds which I have considered (Econometrica's paper, p. 714). The conclusion is that the bond rate is not a good indicator of the pure rate of interest.

Thus, FISHER's objection might be valid if, and only if, we know the right value of i. But we don't know the right value of i and the errors on i can explain the greater range of variation of Θ_a.

In response to Professor FISHER, I must again stress that I did not say that the facts prove the validity of my model, but only that my model is in agreement with every known fact (§ 337 of my paper), and nothing else.

HAAVELMO

I think we ought to distinguish clearly between what you say about the shape of the production function and what you say about the capital-output ratio, because the latter does not depend on the production function alone. It depends on the whole pattern of economic behaviour, that is, on many other things besides the production function.

MALINVAUD

In his very stimulating contribution, Professor ALLAIS uses a model of capital theory in which primary inputs are introduced continuously and « mature » in final outputs after some time, the delay, between input and output being subject to a fixed or varying distribution. Such a model, that may be traced back to JEVONS, is rather neglected today. I am glad to take this opportunity to express the view that we should probably use that model more than we in fact do. It has rendered great services to capital theory in the past, and I am sure it can still render great services in the future. This is well illustrated by Professor ALLAIS' results.

ALLAIS

As far as the HAAVELMO objection is concerned, I would say that naturally every model involves certain simplifications. I do not say that this model can represent absolutely the whole of reality. But the hypotheses of the model seem to be quite appealing and they are justified by the empirical research undertaken. Now the conclusions of the model are in complete agreement with facts Thus, perhaps we can accept this theory at least as a provisional conclusion.

[11] *Allais* - pag. 306

SPATIAL ORGANIZATION AND REGIONAL PLANNING: SOME HYPOTHESES FOR ECONOMETRIC ANALYSIS

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1. Introduction

The objective of this paper is to raise certain fundamental questions regarding the application of existing econometric and regional science techniques to the problems of regional planning. It is my hope that by raising these questions, I will be able to stimulate the formulation of hypotheses which can be tested with new kinds of data. Perhaps these hypotheses may then lead to more effective theories and techniques with reference to comprehensive planning for economic development.

The basic issue which I wish to confront is one which has been avoided by econometricians and regional scientists. It is: what are the properties of an optimal spatial allocation of decision-making authority, planning functions, and other selected governmental functions (inclusive of certain aspects of administration)? Alternatively, for an hierarchical system of regions, what should be the locational distribution of such authority and functions? Furthermore, to what extent will the above spatial allocation affect and be affected by the spatial distribution of industrial production, population, investment, consumption and other economic magnitudes? The latter question suggests the need to develop a more adequate general interregional theory which encompasses the spatial distributions of both economic activities and political-administrative-planning functions and decision-making authority; and that these spatial distributions be simultaneously determined (1).

In what follows, I can only make a small beginning at the formidable task which confronts us. I shall suggest hypotheses which relate to:

1) the advantage (positive or negative) of *increased participation potential* with increase in the degree of spatial decentralization of decision-making authority (utilizing concepts based upon the gravity model as developed in the field of regional science);

2) the advantage (positive or negative) of increase in the degree of spatial decentralization with respect to:

   a) cost of collecting information

   b) cost of processing information

   c) cost of transmitting information

   d) time-cost of executives, representatives and officials spent in reaching a decision based upon the processed information made available;

3) the *overview* advantage (positive or negative) of increase in the degree of spatial decentralization (where *overview* advantage is defined in terms of the wisdom-full information-coordination factor in decision making as reflected in the writings of Professor J. MARSCHAK and his associates).

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(1) For further elaboration of these question and for detailed exposition of certain concepts utilized in this paper, the reader is referred to W. ISARD and T. TUNG: *Some Concepts for the Analysis of Spatial Organization*, « Papers and Proceedings of the Regional Science Association », Part. I, Vol. 11, and Part. II, Vol. 12. Permission to reproduce without change many of the statements of these two manuscripts is hereby acknowledged.

2. Participation potential and degree of spatial decentralization

Elsewhere I have proposed to measure participation potential for a system of n nodes (points, central places, regions) as follows:

$$_iV = \sum_j {}_iV_j = \sum_j G_{ij} \frac{w_j M_j^\beta}{d_{ij}^b}$$

where d_{ij} = distance between nodes i and j, however defined, $i, j = 1, \ldots, n$

M_j = mass at j, however defined;

b and β = adjustment factors, whether constant or variable, applying to d_{ij} and M_j, respectively;

w_j = a weight, however constructed, to be applied to M_j;

G_{ij} = an appropriate constant applicable to potential interaction between nodes i and j;

$_iV_j$ = participation potential of individuals (mass) at j per unit decision to be reached at i; and

$_iV$ = total participation potential of all nodes in the system per unit decision to be reached at i.

Now consider a hierarchy which may be defined by several levels or orders of nodes. (Assume h orders of nodes, an order being indicated by f or g; $f, g = 1, \ldots, h$). In a simple hierarchy, the single first-order node may be represented by the peak of the pyramid or the top of the tree. An h^{th} order node is one of the many at the base of the pyramid or tree. Intermediate nodes may exist at one or more intermediate levels or orders. The hierarchy may be regularly or irregularly structured, and may be symmetrical or asymmetrical. The flow of participation or the exercise of influence may be upward throughout, downward throughout, or both ways throughout; or it may be characterized by any one of the many possible combinations of upward and downward movements.

Take the symmetrical tree-like hierarchy of Figure 1. Each node, starting from the peak, leads to four subordinate nodes. The tree consists of four orders, the number of nodes of the

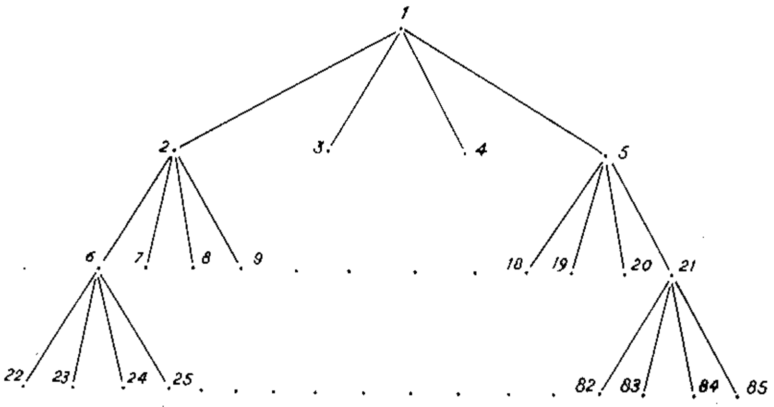

FIG. 1 — *A regular Hierarchy of Nodes*

first, second, third and fourth-orders being 1, 4, 16 and 64, respectively. Their numbers run from 2 to 5 in the second-order, 6 to 21 in the third, and 22 to 85 in the fourth. At the start, let there be one person at each node, each person having the same weight of unity. We also set $d_{ii} = \varepsilon = \frac{1}{10}$ and let G_{ij}, β, $b = 1$. When $d_{\overline{fg}}$ represents the distance between two nodes of orders f and g along any given branch from the peak (the bar being placed over the subscript to indicate that we are speaking about the order of node), $d_{\overline{fg}} = 1$ when $f > g$; and $d_{\overline{fg}} = \infty$ when $f < g$. The sense of the last assumption is that only upward flow of participation is permitted. A person cannot participate in decisions at any node of higher order than the one at which he is located.

Participation Potential may be computed for various spatial patterns of decision-making authority (with reference to a defined set of decisions). If we normalize these patterns, so that in each case the total authority is represented by unity, we are in a better position to carry through the calculations [2]. In table 1 different degrees of spatial decentralization are indicated by the several rows, row A representing 100% centralization and row Z representing a very high degree of, if not 100%, decentralization. Here, we do not attempt to measure in absolute terms the degree of spatial decentralization corresponding to any allocation among nodes of decision-making authority. Rather, we can state that according to certain readily accepted criteria, some patterns are more spatially decentralized than others. For example, A, B, C, D, E, F, P, Q, S, X and Z is an ordering of patterns corresponding to increasingly higher degrees of spatial decentralization [3]. On the other hand, there are no readily accepted criteria by which one can state that of patterns H or G (or patterns S or T), one or the other involves greater decentralization. Row A represents the highest degree of centralization of decision-making authority, all decisions being made at the single first-order node. Zero amount of decisions are made at each 2nd-order, 3rd-order and 4th-order node. Row B represents a somewhat less centralized situa-

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(2) This step precludes any effect of the variable, the spatial pattern of decision-making authority, upon the total amount of authority that may exist. (We may implicitly assume that the optimal total amount for any given organization has already been determined, or that the total amount is prescribed beforehand). In a more general statement this effect should be encompassed.

(3) The readily accepted criteria are: 1) if there are two patterns different with respect to the amount of decision-making authority at two orders of nodes only, then the one having the larger amount of decision-making authority at the higher order node is the more decentralized of the two; 2) if there are two patterns different with respect to three or more orders of nodes, and if there is still another pattern which according to criterion (1) is more decentralized than one of the two patterns but less than the other, then the latter is the more decentralized of the two; and so forth.

TABLE 1

Participation Potential for Patterns of Spatial Decentralization

Pattern of Spatial Decentralization	Node of First Order	Nodes of Second Order	Nodes of Third Order	Nodes of Fourth Order	Total Participation Potential
	NODE NUMBER				
	(1)	(2) ... (5)	(6) ... (21)	(22) ... (85)	P
A	1	0 ... 0	0 ... 0	0 ... 0	43.33
B	.9	.1 1	0 ... 0	0 ... 0	47.83
C	.8	.2 2	0 ... 0	0 ... 0	52.27
D	.75	.25 ... .25	0 ... 0	0 ... 0	54.50
E	.75	.20 ... .20	.05 ... .05	0 ... 0	61.30
F	.6	.2 2	.2 2	0 ... 0	88.40
G	.6	.2 2	.1 1	.1 1	130.00
H	.5	.5 5	0 ... 0	0 ... 0	65.67
I	.5	.25 ... .25	.25 ... .25	0 ... 0	99.67
J	.5	.25 ... .25	.125 ... .125	.125 ... .125	151.67
K	.5	.125 ... .125	.25 ... .25	.125 ... .125	168.67
L	.5	.125 ... .125	.125 ... .125	.25 ... .25	220.67
M	.4	.6 6	0 ... 0	0 ... 0	70.13
N	.4	.4 4	.2 2	0 ... 0	97.33
O	.4	.3 3	.2 2	.1 1	152.53
P	.4	.2 2	.2 2	.2 2	207.73
Q	.4	.2 2	.1 1	.3 3	249.33
R	.4	.1 1	.2 2	.3 3	262.93
S	.4	.1 1	.1 1	.4 4	304.53
T	.3	.4 4	.3 3	0 ... 0	115.40
U	.3	.3 3	.3 3	.1 1	170.60
V	.3	.3 3	.2 2	.2 2	212.20
W	.3	.2 2	.2 2	.3 3	267.40
X	.3	.1 1	.1 1	.5 5	364.20
Z	0	0 ... 0	0 ... 0	1.0 ... 1.0	640.00

tion. For every branch, the fraction of all decisions made at the single first-order node is 0.9. Each 2nd-order node makes 0.1 of all decisions on any branch on which it is located. Each 3rd-order and 4th-order node makes zero fraction of all decisions. In the situation represented by Q, 0.4 of all decision-making authority resides at the top; 0.2 of all decision-making authority along any branch resides in the 2nd order node of that branch; 0.1 of all decision-making authority along any branch resides in the 3rd-order node of that branch; and 0.3 of all decision-making authority along any branch resides in the 4th-order node of that branch. At the bottom of table 1, row Z represents the case where all decisions for any branch are made at the 4th-order node of that branch, none being made at any node of any other order. Note that, because the flow of participation and exercise of influence can only proceed upward, no node can influence decisions at any other node of the same or higher order.

Given our assumptions and this framework we compute total participation potential, P, of all individuals over all decisions for each of the situations of table 1. By definition, in our normalized cases, $P = \sum_{i=1}^{n} r_i \cdot {}_iV$, where r_i is the fraction of all decisions to be made at node i, this fraction being the same for all nodes of the same order as i (4). The values for P are re-

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(4) If there are s spatial patterns of decision-making authority to be considered, each corresponding to a degree of spatial decentralization, then for our set of assumptions the participation potential corresponding to each degree is given by the single row vector $[P_q]$:

$$[P_q] = [k^{g-1}] \cdot \left[\frac{1}{d_{\overline{fg}}}\right] \cdot [\tau_{gq}] \qquad \begin{matrix} q = 1, \dots, s \\ f,g = 1, \dots, h \end{matrix}$$

where: k is the constant defining the number of nodes in the next higher order directly linked to any given node; $[k^{g-1}]$, $g = 1, \dots, h$ is a single row vector $1 \times h$, h being the number of orders in the hierarchy; $d_{\overline{fg}}$ is the distance between a node of order f and the node of order g to which it is most directly connected; $\left[\frac{1}{d_{\overline{fg}}}\right]$ is an $h \times h$ matrix (f, $g = 1, \dots, h$); and $[\tau_{gq}]$ is a $h \times s$ matrix, each column of which describes a spatial pattern corded in the last column of table 1. As can be expected to some extent at least, participation potential increases with increase in degree of spatial decentralization. This is the case for rows A to G where each row involves greater spatial decentralization than the preceding row. When we come to row H it is no longer clear, as indicated, whether G or H involves a higher degree of spatial decentralization. Note that P falls from 130.00 for G to 65.67 for H. From row H to and including row L, there is once again a succession of patterns, each one involving a higher degree of spatial decentralization than the preceding one. The value of P also rises, without exception, from one pattern to the next. At row M, once again it is not possible to state that the pattern has a higher degree of spatial decentralization than the preceding one (row L). We also note that P at M is lower than P at L.

(To illustrate the dependence of P on the choice of values for the basic parameters, we carry through several more computations. The results are given in table 2. In column 1 of

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or allocation of decision-making authority among the modes in the h order hierarchy. In the calculation of the participation potentials of the last column of table 1, $d_{\overline{fg}} = \infty$, whenever $f < g$ since a downward flow of participation or exertion of influence has been precluded by assumption, and $d_{ii} = \varepsilon$ which we set equal to 1/10. The value of h is 4 so that the $[h^{g-1}]$ vector is [1, 4, 16, 64]. The $\left[\frac{1}{d_{\overline{fg}}}\right]$ matrix becomes:

$$\begin{bmatrix} \frac{1}{1/10}, & \frac{1}{\infty}, & \frac{1}{\infty}, & \frac{1}{\infty} \\ \frac{1}{1}, & \frac{1}{1/10}, & \frac{1}{\infty}, & \frac{1}{\infty} \\ \frac{1}{2}, & \frac{1}{1}, & \frac{1}{1/10}, & \frac{1}{\infty} \\ \frac{1}{3}, & \frac{1}{2}, & \frac{1}{1}, & \frac{1}{1/10} \end{bmatrix}$$

The product of $[h^{g-1}]$ and $\left[\frac{1}{d_{\overline{fg}}}\right]$ is the row vector [43 1/3, 88, 224, 640]. This row vector, when multiplied by the matrix $[\tau_{gq}]$ yields a row vector which is the transpose of the last column of table 1.

table 2 are reproduced the figures of the last column of table 1. In column 2 are recorded the figures for the case as characterized at the top of column 2 (which case is the same as that of column 1 except that $d_{\overline{21}}$ is taken to be equal to 2 instead of unity). In columns 3 to 7 are recorded the figures for several other cases, each case being characterized at the top of the respective column.

The computations for many other cases might be made, for example cases involving different values for ε, β, b, G_{ij}, and different assumptions on the number of individuals at each node (all the cases of table 2 assume only one individual at each node). These the reader may do for himself (5).

We now wish to proceed to the next step, namely, the transformation of values for participation potential into values for average productivity, i.e. the mapping of a participation potential space onto an average productivity space.

Since we are unaware of any comprehensive empirical work on this problem, we can only pose the question of the validity of certain hypotheses which seem reasonable in the light of social science knowledge. Can we state a certain functional relationship between average productivity (dependent variable) and P (independent variable) which can be represented by a curve such as one of the curves on Figure 2 (6)? Starting with the extreme at which P is at a minimum (which tends to correspond to the pattern of 100% spatial centralization), can

(5) He may also perform calculations for irregular tree structures as well as for regular tree structures where the number of nodes at any given order (except the first) is other than a k multiple of the number of nodes at the next lower order.

(6) As previously noted, we do not treat explicitly the effect of P upon the total amount of decision-making authority (which may be viewed as control), and vice-versa. In a more general framework, this effect would be recognized; and the influence of the variable, total amount of decision-making authority, upon productivity would be recognized. In this connection, see ARNOLD S. TANNENBAUM, *Control and Effectiveness in a Voluntary Organization*, « American Journal of Sociology », Vol. 67, July 1961, pp. 33-46, and *Control in Organizations: Individual Adjustment and Organizational Performance*, « Administrative Science Quarterly », Vol. 7, September 1962, pp. 236-257.

TABLE 2

Total Participation Potential for Different Values of Basic Parameters

Pattern of Spatial Decentralization	G, α, β, b, = 1						
	$w_1 = w_2 = w_3 = w_4 = 1$				(5)	(6)	(7)
	(1) $d_{\overline{21}}, d_{\overline{32}}, d_{\overline{43}} = 1$ $d_{\overline{12}}, d_{\overline{23}}, d_{\overline{34}}, = \infty$ $d_{ii} = 1/10$	(2) same as col. (1) except $d_{\overline{21}} = 2$	(3) same as col. (1) except $d_{\overline{21}} = 2$ $d_{\overline{43}} = 1/2$	(4) same as col. (1) except $d_{\overline{12}}, d_{\overline{23}}, d_{\overline{34}} = 1$	$w_1 = 3$ otherwise same as col. (4)	$w_1 = 3$ $w_2 = 2$ $w_3 = 1$ $w_4 = 1/2$ otherwise same as col. (4)	same as col. (6) except $d_{\overline{13}}, d_{\overline{24}}$, $d_{\overline{14}} = \infty$
A	43.33	33.33	35.61	43.33	63.33	56.67	56.67
B	47.83	38.80	41.92	47.90	66.10	62.50	62.50
C	52.27	44.27	48.22	52.46	68.86	68.33	68.33
D	54.50	47.00	51.39	54.75	70.25	71.25	71.24
E	61.30	53.80	60.83	61.73	77.18	75.58	75.49
F	88.40	82.40	98.70	89.50	102.10	97.30	97.00
G	130.00	124.00	133.90	132.48	145.05	111.25	110.60
H	65.67	60.67	67.15	66.16	77.17	85.83	85.83
I	99.67	94.67	114.48	101.04	111.80	107.46	107.08
J	151.67	46.67	158.48	154.77	165.49	124.89	124.08
K	168.67	163.67	182.14	172.22	182.81	135.72	134.71
L	220.67	215.67	226.14	225.93	236.49	153.16	151.71
M	70.13	66.13	73.46	70.73	79.93	91.66	91.66
N	97.33	93.33	111.30	98.63	107.63	108.96	108.66
O	152.53	148.53	165.44	155.56	164.43	131.56	130.76
P	207.73	203.73	219.57	212.50	221.23	154.16	152.86
Q	249.33	245.33	254.77	255.48	264.18	168.11	166.46
R	262.93	258.93	273.71	269.43	278.03	176.76	174.96
S	304.53	300.53	308.91	312.42	320.98	190.71	188.56
T	115.40	112.40	136.54	117.15	124.24	123.45	123.00
U	170.60	167.60	190.68	174.08	181.05	146.05	145.10
V	212.20	209.20	225.88	217.07	224.00	160.00	158.70
W	267.40	264.40	280.01	274.00	280.80	182.60	180.80
X	364.20	361.20	369.35	373.92	380.55	219.15	216.50
Z	640.00	640.00	640.00	658.33	659.00	341.00	336.00

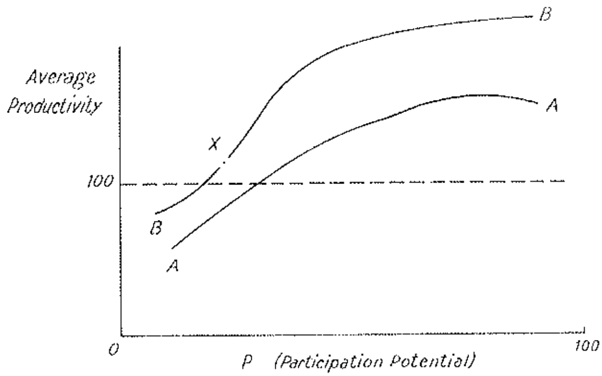

FIG. 2

we hypothesize that average productivity increases with increases in P (and presumably increase in spatial decentralization)? Would this increase of average productivity in the early stage be at an increasing rate (curve BB) or decreasing rate (curve AA)? If it is at an increasing rate, can it be presumed to reach a point where it is at a constant rate (point X, curve BB) and then a decreasing rate with still greater values for P (and presumably still higher degrees of spatial decentralization)? Further, at some later point, at a relatively high value for P (and presumably a relatively high degree of spatial decentralization), can average productivity be posited to fall off (as in curve AA)? (7).

(7) For some relevant empirical findings, See T. TOMEKOVIC, *Level of Knowledge of Requirements as a Motivational Factor in the Work Situation,* « Human Relations », Vol. 15, No. 3, 1962, pp. 197-216; J.R.P. FRENCH, Jr., J. ISRAEL and DAGFINN ÅS, *An Experiment on Participation in a Norwegian Factory,* « Human Relations », Vol. 13, No. 1, February 1960, pp. 3-19. It is recognized of course that other variables are relevant. For example, on the effect of personality upon the relation between participation in decision making and productivity, see VICTOR H. VROOM, *Some Personality Determinants of the Effects of Participation,* « The Journal of Abnormal and Social Psychology, Vol. 59, November, 1959, pp. 322-327. Also see RALPH E. DAKIN, *Variations in Power Structures and Organizing Efficiency: A Comparative Study of Four Areas,* « The Sociological Quarterly », Vol. 3, July 1962, pp. 228-250; and RENSIS LIKERT, *A Motivation*

Just as in demand curve estimation, there are many complex problems in the estimation of an average productivity-participation potential curve. We cannot discuss these problems here. Once an average productivity-participation potential curve is estimated, for certain economic-type organizations, can such a curve be converted into an « extra-returns » curve? Can a total output curve be constructed where total output of the organization is in part a function of P? When price of the output is unaffected by changes in total output associated with variation in P, can such changes be multiplied by the constant price to determine the extra-returns (positive or negative) corresponding to each value of P (relative to some base value of P)? When price of the output falls with increase in total output ,then, given a price-output function, can change in total revenue and thus extra-returns for any increase in P be determined? With reference to a defined set of decisions or to decisions on a concrete issue or set of concrete issues, can the extra-returns curve be represented, for example, by a curve such as curve CC in Figure 3? (8).

Approach to a Modified Theory of Organization and Management, and CHRIS ARGYRIS, *Understanding Human Behavior in Organisations: One Viewpoint,* both in Mason Haire (ed), « Modern Organization Theory », John Wiley, New York, 1959.

As VROOM states, after reviewing the literature:

« When the entire pattern of results is considered, we find substantial basis for the belief that participation in decision-making increases productivity. There is both experimental and correlational evidence indicating that high levels of influence by workers in making decisions that they are to carry out result in higher productivity than lower levels of influence. It should be noted, however, that not all the findings are consistent with this generalization. The results of both LEWIN, LIPPITT and WHITE (1939) and MORSE and REIMER (1956) suggest that, under some conditions higher productivity may be achieved with use of more autocratic methods ».

VROOM urges careful investigation of the relationships between changes in average participation in decision making and resulting changes in the quality of decisions made, the strength of group standards regarding execution of the decisions and the worker's « ego involvement » in the decisions. (See VICTOR R. VROOM, *Work and Motivation,* John Wiley, New York (forthcoming).

(8) Note that in the hierarchical tree-structure which has been assumed, lateral communication is precluded. However, it may be hypothesized that at least up to a point lateral communication is highly desirable and has a

For non-economic or hybrid organizations, the question of the conversion of an average productivity-participation potential curve into an extra-returns curve is still more difficult. For a system planning bureau, having jurisdiction over all local, regional and national planning, change in average productivity may be associated with change in the quality of the service it provides (as measured by the effectiveness of its planning for

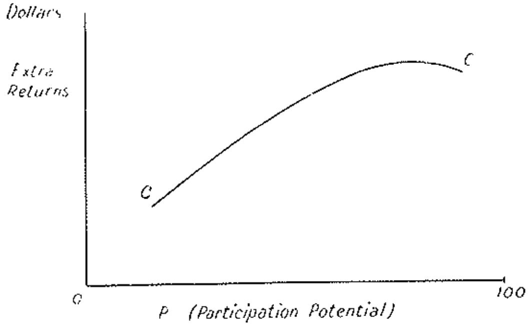

FIG. 3

social welfare). But, as we all know, and as is the case for many economic, non-economic and hybrid organizations, it is exceedingly difficult to measure changes in quality, to price

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positive effect upon organization maintenance. (See JAMES E. MCNULTY, *Some Economic Aspects of Business Bureaucracy*, Wharton School, University of Pennsylvania, 1962, mimeographed, pp. 43-53.) In this connection, MCNULTY and others have suggested the following relationships:

1) when there is a highly centralized decision-making structure and all flows are vertical, lateral communication (and thus morale and average productivity) is relatively low;
2) with decreases in centralization there tends to be increase in lateral communication (and thus morale and average productivity);
3) however, when the decision-making structure tends to be highly decentralized (approaches local autonomy) lateral communication (and hence morale and average productivity) may tend to decline.

Consequently, the « extra-returns » function associated with participation potential is to be adjusted for the lateral communications factor when this factor is a variable for a set of possible decision-making structures.

different qualities of a service, and thus to estimate an extra-returns curve (9).

Where an organization is conceived to be society itself, can change in average productivity of its labor force be translated into changes in Gross System Product (inclusive of non-economic type commodities appropriately priced)? Can an extra-returns curve be constructed to relate changes in Gross System Product to changes in participation potential?

3. Information, communication and other decision-making costs

In the consideration of the spatial decentralization of decision-making authority for an organization, it is important to identify the differences among patterns in cost of information collection, processing, and transmission, and of executive time and other items. Unfortunately, this is another area in which the accumulation of empirical materials and empirically-based hypotheses is inadequate (10). Nonetheless, it is essential to treat explicity these costs. In suggesting hypotheses, we shall draw heavily upon some of the pioneering thinking of J. Marschak and T. Marschak (11).

(9) Alternatively, extra-returns may be conceived as psychic income, or good will, or political income based upon power, prestige, respect, affection and other Lasswell-type commodities.

(10) There are, however, interesting and somewhat, related materials on administrative costs, such as in James McNulty, *Administrative Costs and Scale of Operations in the U.S. Electric Power Industry - A Statistical Study*, « The Journal of Industrial Economics, Vol. 5, November 1956, pp. 30-43; P.G. Herbst, *Measurement of Behavior Structures by Means of Input-Output Data*, « Human Relations », Vol. 10, No. 4, 1957, pp. 335-346; and Theodore R. Anderson and Seymour Warkow, *Organizational Size and Functional Complexity: A Study of Administration in Hospitals*, « American Sociological Review », Vol. 26, February 1961, pp. 23-28. Also see William R. Hughes, *Short-Run Efficiency and the Organization of the Electric Power Industry*, « Quarterly Journal of Economics », Vol. 76, November 1962, pp. 592-612.

(11) See the works of Jacob Marschak already cited and Thomas Marschak, *Centralization and Decentralization in Economic Organizations*, « Econometrica », Vol. 27, July 1959, pp. 399-430.

For the purposes of this paper we shall consider four basic cost components. The reader, of course, may wish to hypothesize a smaller or greater number of cost components, and employ a different set of definitions. Our four basic cost components are:

a) cost of collecting information, where each node collects information on conditions in its own tributary (jurisdictional) area ([12]);

b) cost of processing and beneficiating information (inclusive of computation costs and all research costs not elsewhere covered);

c) cost of transmitting (transporting) information from one node to another;

d) cost of the time of the executive, local representative and other officials in reaching decisions individually or in conference ([13]).

We now set forth, in the form of questions, some reasonable hypotheses concerning each of these components ([14]).

Cost component a): Can it be presumed that: 1) most of the information needed for decision-making is on local condi-

([12]) Thus an h^{th} order node collects information on its local area, whereas the single 1st-order node collects information that pertains to the system of nodes (regions) as a *system*.

([13]) At times, several or all of these cost items may be considered as administrative cost, and less frequently as cost of « routine » administration. We prefer to consider any administrative cost not covered by our classification as belonging to the broad category of production costs.

Differences in costs which may arise from different degrees of articulation and coordination of activities associated with different patterns of spatial decentralization will be treated later as differences in « overview advantage. »

([14]) It should be kept in mind that each of these components may be directly related to various explicit rules that may be used in determining the volume and nature of information to be collected, the ways in which information is to be processed, the type and volume of information to be communicated, the media by which communications are effected, the kind of information to be considered by decision makers, the form of its presentation and so forth. Clearly, these rules will be related to attitudes and other variables which are to be considered in a later section.

tions and must be obtained at local nodes; 2) a large fraction of such information is visible to local people; and 3) the visibility to non-local people of such information on a local node rapidly falls off with distance from that node? If so, can we conclude that the greater the distance a decision-making point is from a given local node, the greater the volume of information on that node and its tributary area which must be formally collected and put in explicit form, *ceteris paribus*? Can we hypothesize that for decisions on a representative concrete issue or set of concrete issues cost component *a*) is some monotonically increasing function of the degree of spatial centralization as suggested by the curve of Figure 4? (15).

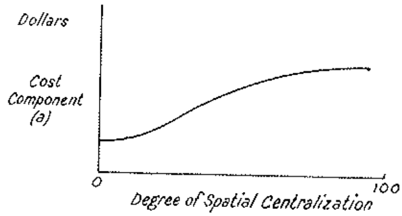

FIG. 4

Cost component b): When no slack facilities, labor and other resources exist, and when inputs are costed in full at prevailing rents and wage rates, do major scale economies obtain in processing and beneficiating information? For processing a given volume of information associated with decisions on a concrete issue or set of concrete issues, can we hypothesize that cost component *b*) is some monotonically decreasing func-

(15) In this function, should allowance be made for the likelihood that important information pertaining to any f^{th}-order node and its tributary area may be more visible to a person at the f^{th}-order node than to a person at an $(f+1)^{th}$ or higher order node?

Note that the functions underlying the curve of Figure 4 and the curves of subsequent figures are taken to be continuous. In practice, they are likely to involve steps and other discontinuities. The argument, however, remains unaffected.

tion of the degree of spatial centralization-as suggested by curve M in Figure 5? ([16]). However, if one postulates that at each local and regional node there exists at least some supply of zero-cost decision-making labor, as is frequently the case in the literature on socialist economics and centralized planning, might the relevant function be represented by curve N? ([17]) Can the spread between curves M and N at any point along the horizontal axis be taken to represent the cost savings from zero-priced computing and beneficiating resources for the corresponding degree of spatial centralization? Further, if it is accepted that local decision makers have greater local visibility, and therefore need to collect less information, can we hypothesize that their need for computation and data processing is accordingly smaller? Would the dotted curve Q be relevant rather than curve N?

Cost component c): In analyzing the costs of transmitting information, is it valid to consider only two elements: terminal costs (inclusive of coding and decoding costs) and operating or « line-haul » costs which are usually a direct function of intervening distance? Of course, both these costs should be taken to cover maintenance expenses and fixed charges on investment in communication (transportation) channels ([18]). When operating costs rise less than proportionately with distance, can the function relating transmission cost per standard unit

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([16]) After a point, it may also be hypothesized that scale diseconomies arise because of congestion, overload of facilities, etc.

([17]) An advantage typically claimed for decentralized planning of the economy is a personnel saving because the managers at each local node are « a ready-made computational staff and all of them are required in any case to be employed for the tasks of managerial supervision. In their remaining time they simultaneously perform all the tasks of computing each revised set of production decisions except for the computation of prices (which is performed by the central agency). In the centralized solution, on the other hand, the entire burden of these computational tasks falls on the central agency, and the agency must hire a staff especially for this purpose... » (T. MARSCHAK, *op. cit.*, p. 400).

([18]) Parallels to these cost elements may be taken to exist when information is transported by individuals.

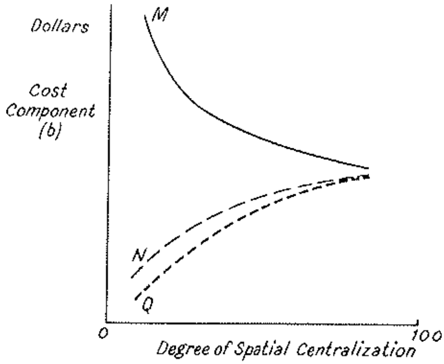

FIG. 5

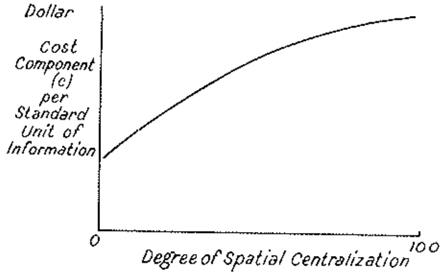

FIG. 6

of information to degree of spatial centralization be depicted by a curve such as that of Figure 6 (where the y intercept is taken to be terminal costs per standard unit of information)? (19) Can the product of (1) the function underlying the curve of Figure 6 and (2) the function relating the total volume of information collected to degree of spatial decentralization

(19) Can it be posited that *on the average* the higher the degree of spatial centralization, the longer the distance of transmission of a standard unit of information? We should recognize of course that certain units of information, for example, that generated at the single, 1st-order node, need to be transmitted less with increase in degree of spatial centralization.

(which is fundamental to the construction of the curve of Figure 4) yield the relevant (total) function for cost component *c*) as suggested by the curve of Figure 7?

Cost component d): Although the cost of the time spent by decision makers in reaching decisions might be aggregated with cost component *b*), for analytical purposes we prefer to keep these two cost items separate. As with cost component *b*), if such time is fully costed, may we presume that significant scale economies exist? Would the number of times at which a decision on a concrete issue (or set of concrete issues) is to be taken

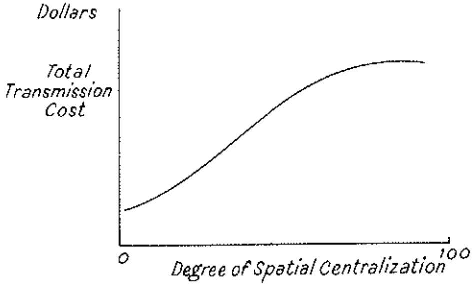

FIG. 7

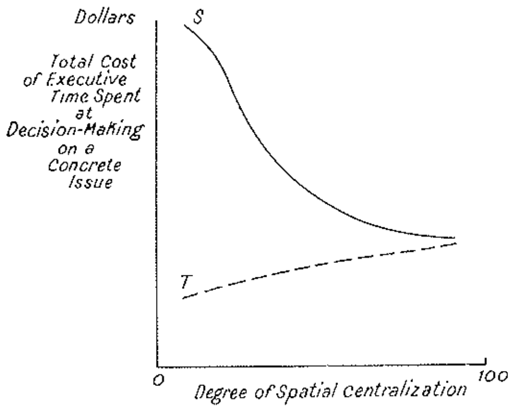

FIG. 8

be the greater, the smaller the degree of spatial centralization? Thus, would total time spent in reaching decisions on a concrete issue be the greater, the smaller the degree of spatial centralization? Could the relevant cost function of executive time be depicted by a curve such as curve S of Figure 8 after adjustment for different levels of efficiency and prices of executives at different order nodes? On the other hand, if a supply of slack executive (decision-making) labor is assumed to exist at each node and if this labor is assigned a zero cost, then might curve T of Figure 8 be considered relevant?

It is clear from the preceding discussion that a significant amount of empirical research must be conducted to permit the development of firm hypotheses on information and communication costs within an organization. Pending such research we may, *for pedagogical purposes alone,* set down a total cost curve with reference to decisions on a representative concrete issue (or set of concrete issues). If we postulate that no slack resources exist and that all labor and facilities are properly priced, we may let the curve in Figure 9 depict the relevant total cost function. Other curves of different slope and also of positive slope may of course be considered *equally* valid.

4. Overview advantage (and disadvantage)

We now turn to another set of significant factors which vary with the degree of spatial decentralization. These factors relate to the ability to make wise or good decisions. It has generally been claimed that the decision-maker (the individual, planning bureau, or government agency) at the 1st-order node is able to make better decisions than others, *ceteris paribus*. However, does the empirical evidence support this claim? Within the tree-like organization of the preceding paragraphs, does the decision maker at the 1st-order node have available a much greater amount of relevant information than decision makers at local nodes? [20] Is he in a better position to reach a more consistent, or coordinated, or articulated set of decisions pertaining to a representative concrete issue (or set of concrete issues) than are the decision makers at higher-order nodes, who at least to some extent act independently of each other? Can he be said to have an overview advantage in decision making? Thus, can we hypothesize that the more an organization's decisions are made by the decision maker at the first-order node

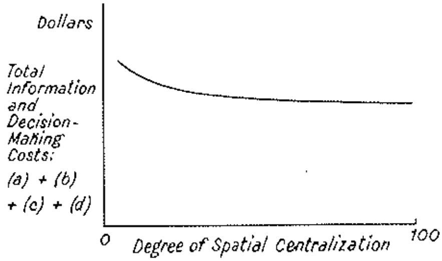

FIG. 9

(or by decision makers at lower-order nodes), in general the better the resulting set of decisions, *ceteris paribus?* In short, can an overview advantage be said to exist, which varies directly with the degree of spatial centralization in the decision-making structure of an organization? [21]

There have been several hypothetical, highly simplified examples presented to illustrate the nature of this overview advantage. MARSCHAK has developed the case of a ship-build-

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([20]) In raising this question one must also keep in mind the extent to which information may flow from lower-order nodes to higher-order nodes, and from one node along one branch to another node of the same order along a different branch via a lower-order node.

([21]) For some relevant discussion see J. MARSCHAK in Haire, *op. cit.*; BLAU and SCOTT, *op. cit.*, pp. 121-128; MARCH and SIMON, *op. cit.*, pp. 201-210; C. B. MCGUIRE, *Some Team Models of a Sales Organization*, « Management Science », Vol. 7, January 1961, pp. 101-130; ROY RADNOR, *The Application of Linear Programming to Team Decision Problems*, « Management Science », Vol. 5, January 1959, pp. 143-150.

ing firm acting as a team and confronted with the possibilities of either a centralized system of decision-making, or a decentralized system. (22) His resulting figures would not be inconsistent with the hypothesis which states that an overview advantage does exist, and with the more concrete hypothesis that states that this advantage can be roughly depicted by curve CC of Figure 10. [In Figure 10, curve CC indicates a disadvantage (a negative Figure) for any degree of spatial centralization less than 100%]. The present author together with Tung has developed another example involving the investment by a large merchandising firm of $1 million at each of several h^{th} order nodes (23). The resulting figures are also not inconsistent with the hypotheses underlying curve CC of Figure 10.

Clearly, neither of these two examples can be said to support in any way whatsoever any hypothesis on overview advantage. A considerable amount of spade work has yet to be done in defining concepts and terms before any empirical

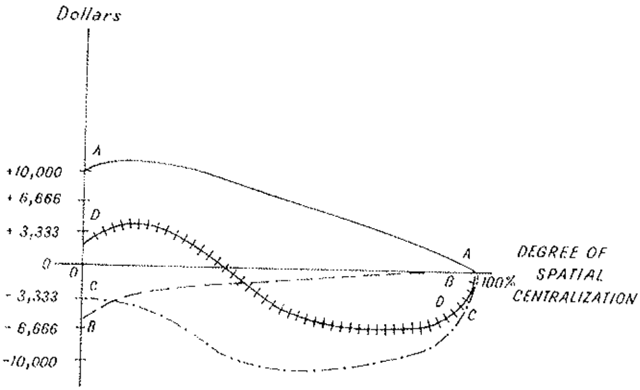

FIG. 10

(22) MARSCHAK, *op. cit.*
(23) ISARD and TUNG, *op. cit.*, Part. II.

test can be undertaken. Nonetheless, for pedagogical purposes we shall utilize curve CC of Figure 10 to illustrate overview advantages for different degrees of spatial decentralization.

5. The net effect of participation potential, information-communication-decision costs and overview advantage

We now examine the interplay of the several factors examined, where only one state of the environment can occur. Such examination immediately implies the hypothesis that the factors examined are in effect the relevant factors. This hypothesis is to be seriously questioned. However, if one does accept it, and if one accepts the hypotheses implied by the extra-returns curve of Figure 3, then in Figure 10 curve AA may be taken to represent extra-returns with respect to decisions on a representative concrete issue; however, the horizontal axis is taken to measure degree of spatial centralization (rather than participation potential). Also the pattern of 100 percent spatial centralization is taken to be a point of reference for purposes of comparison; its extra-returns is set at zero.

Further, if one were to accept the hypotheses underlying the curve of Figure 9, then in Figure 10 curve BB depicts variation in total information, communication and decision-making costs, again where this cost at 100 percent spatial centralization is taken to be zero. Curve CC, as already indicated, may be taken to portray overview advantage (disadvantage) where such advantage is set at zero at 100 percent centralization.

Since all advantages, disadvantages, and extra-returns for the pattern of 100 percent spatial centralization are set at zero, may it be hypothesized that the construction of a fourth curve is meaningful? This curve, curve DD of Figure 10, at each point of the horizontal axis indicates for the corresponding pattern of spatial centralization the net advantage (disadvantage) of this pattern relative to the pattern of 100 percent spatial centralization. Where curve DD lies below the horizontal axis, there is a net disadvantage for the corresponding pattern. Where DD lies above the horizontal, there is a net advantage. Obviously, where the entire set of hypotheses underlying he curves of Figure 10 is accepted, the optimal degree of spatial centralization would correspond to that point at which curve DD reaches a maximum.

The curves of Figure 10, and the discussion thus far, relate to decision making on a representative concrete issue (or set of concrete issues) when it is hypothesized that only one, well-defined state of the environment can occur. It is now appropriate to relax this restriction and permit the occurrence of several states of the environment.

Elsewhere, ISARD and TUNG have extended the simple example cited above so that the occurrence of each of three states of the environment is possible ([24]). These states have reference to the international situation and were designated: 1) good will, 2) usual, and 3) extremely tense, as indicated at the head of the columns of table 3. In this example, only overview advantage was permitted to vary with the state of environment for the specific set of assumptions which was adopted. The resulting figures for the case examined are recorded in table 3. In net form, these figures constitute the elements of the payoff matrix of table 4. It is immediately seen that there is no one pattern of spatial decentralization which can be designated optimal. In the light of modern decision-making theory it becomes necessary to introduce still another assumption which pertains to the attitude of the decision-making individual or group ([25]).

([24]) ISARD and TUNG, *op. cit.*, Part. II.

([25]) SEE W. ISARD and M. DACEY, *On the Projection of Individual Behavior in Regional Analysis*, Parts I and II, « Journal of Regional Science », Vol. 4, Nos. 1 and 2, respectively, and articles cited therein.

5. Concluding remarks

The purpose of this paper was simply to confront squarely a basic problem in economic planning for regional development — the problem of an appropriate spatial organization of planning — and other governmental functions, and of the appropriate spatial allocation of decision-making authority. Because this fundamental question has been avoided by econometricians and regional scientists in the past we have been able to present only simple « reasonable » hypotheses. Admittedly, these hypothese are weak, and their empirical testing will lead quickly to new, superior hypotheses. Yet, these hypotheses represent the best that can be culled from the existing voluminous, but rigorless, literature. It is hoped that their presentation will provoke major research effort and contributions to this area.

In closing, it may be mentioned that once empirical testing has been conducted and a number of hypotheses established as valid, it should be possible to estimate decision-making cost differentials for each type of organization, firm, or industry. With such cost differentials on hand, it would then be possible to broaden location analysis in order to embrace the decision-making function. Such broadening would involve essentially the comparison of the decision-making cost differential with the transport cost differential, labor cost differential, and other cost differentials considered as relevant in a location investigation of an organization. The necessary extension of location theory based upon substitution points can easily be achieved. Finally, with a superior classification of organizations, firms or industries as national, regional and local, it would become possible to introduce the decision-making function, at least in part, into one or more of the channels of synthesis of regional techniques which have been described elsewhere ([26]). For example,

([26]) See W. Isard, et al., *Methods of Regional Analysis*, M.I.T. Press. Cambridge, Massachusetts, 1960, Chapter 12.

it would be possible to develop a channel using comparative cost, interregional linear programming, industrial complex techniques centered around a balanced regional input-output model for three or more orders of regions or nodes. Such a channel could be utilized to project not only output, employment, population, income, consumption and investment by region, but also the spatial allocation of decision-making authority and the spatial patterns of decisions, information collection, processing and transmission. However, all this development is for the future. The immediate task ahead is the statement of relatively simple hypotheses in ways that can be rigorously tested against empirical materials.

DISCUSSION

FISHER

I have only a minor comment on Professor ISARD's paper, it is in connection with his use of weights. He uses an expression in which W_i and W_j appear as weights; however, there is nothing about W_i which is specific to i or W_j which is specific to j in that expression, since the form of the expression in fact allows these to be interchanged and the wrong weights applied to the two items. The W_i and W_j therefore ought not to be referred to as weights in the strict sense. On the other hand, ISARD uses these in obtaining a weighted sum of items, the sum being taken over all pairs i and j. When that is done it is the case that W_i appears in every pair involving CD_i and W_j appears in every pair involving CD_j. Thus each pair receives a weight equal to the product of W_i and W_j and it is these products which should be considered as weights. It is thus true that W_i is specific to i and W_j to j although only the pairs are used.

ISARD

Professor FISHER is correct in the context in which he speaks. However, in much of the literature using gravity models, the weights W_i and W_j are used singly, and not in product form. This is the reason I do so throughout my manuscript.

HAAVELMO

I would first like to say that I am sure Prof. ISARD is right in underlining the importance of the kind of problem here mentioned and that also very little has been done about it. Now, the remark I have refers to Prof. ISARD's special model of interaction, and is based on a recent experience I have had. One of my students came to me because he was engaged in some study of retail trade in a regional network of shopping centers. Some of these centers were small and some larger, and the problem was how to locate stores, parking space etc. We got into discussion about the following problem. Consider the interaction formula that you have. In my case it can be regarded as representing how many customers are drawn from i to j and vice versa. We found that the symmetry of this interaction formula was not adequate. We have a passive element, so to speak, and have to distinguish between the active pull and the response. We did not find the final answer, but I think it is a general problem relevant also to your model.

ISARD

I agree that many, if not most situations in the real world involve asymmetry. Many suggestions have been made about what the measure could be in such cases. For any originating node (region) the measure will differ according to type of receiving or terminating point (node). Also, I have presented in my paper a regular hierarchy. Actually, we know that nodes are differently distributed, and that population is of different density in different regions and at different nodes. But to introduce an irregular hierarchy causes all kinds of problems. I do not have the time now to spell out some of the interesting thinking on these problems.

WOLD

When reading this paper in advance, I was quite fascinated by the new vistas it opens up on the analysis of organization problems. At the same time I was puzzled by some of the implications, and I am not quite sure I have grasped the underlying basic hypotheses. This is in particular so with your statement that neither zero nor 100% spatial decentralization work at a disadvantage or at an advantage. Am I right to understand that this result is based on an assumption of complete insight into the future?

To emphasize this question I should like to tell what I once heard from a friend of mine in the foreign service. We had talked about the possibility of predicting developments abont war and peace at the upper decision levels of the great powers, and in particular I had asked whether diplomats find it more difficult to predict such development for the Soviet Union than for the United States. My friend said that the developments are quite unpredictable in both countries, but for entirely different reasons. In the Soviet Union the decisions of the inner circle are watertight secrets, and no outsider can get the smallest clue to predicting what will happen. In the States, he continued, every senator is an open book on his political views, and his opinions can be read in any newspaper; different senators however have different views, and the political decisions are made by committees; what will happen in the future thus is a question of which senators will be members of the committee at issue, and this depends on so many intangible factors that no specialist in the world is able to make valid predictions. Thus although the two systems differ very much with regard to information about the inner circles, the situation with regard to prediction is in reality very much the same, because in the open system the complete insight is only apparent. What is needed for genuine prediction is much more. The point I wish to make is that the assumption of complete insight which is customary in many scientific approaches is indeed very stringent.

FRISCH

I have three suggestions to make. First, we must distinguish between *different things* we may think of when we subdivide according to regions or spatial distribution. For instance we may think of regional *flows of goods and services,* from one region to another, or from one centre to another, inc'uding in this flow of goods and services also the flow of information leading to such concepts as noise in information channels and the like. Second, we may think of the pyramidation, or the regional distribution of *decisional power.*

Third, we must look a bit closer into the question of what we really mean by a region or a center.

I think that when we speak of the first point of view, i. e. the flow of goods and services between regions and centers even including the flow of information, the viewpoint is not fundamentally new. In a sense it does not introduce too much innovation in our way of thinking. But, when it comes to distributing, decisional power, the problem becomes extremely complex. We are facing the difficulty of programming a decisional machinery at the top, and at the same time elaborating rules and regulation for the working of the decisional machinery in the individual regions and centres, which is such that the intentions of the centre are realized when certain decisions and a certain amount of programming analysis is left to be done in the regions and centres. Mathematically speaking, I have found it next to impossible to attack this problem in a straightforward programming way (1) and at the moment I can see no better way out than to build up some kind of simulation or artificially constructed games of decision at various centres. Then, third with regard to the definition — what we mean by a region or a centre. I think it will be a fallacy to concentrate too much on the traditional geographical subdivisions, for instance administrative subdivisions in regions or in states within a union or local administrations within a state and

(1) Note added July 1964: in the spring of this year I made an attempt in a University of Pittsburgh memorandum, building on my non complex method for solving non linear programming problems.

so on. From the viewpoint of a complete programming problem I think we must look first into what I have called the Problem of Patterns of Centres. We may find a certain pattern of centres, which is entirely different from the traditional administrative lines of demarcation. It must be based on an entirely different complex of relations that goes deeper down into the basic problems of economic relations, transportations and so on. This problem of the pattern of centres may be approached somewhat in the same way as the construction of investment projects. In a complete development and investment programming work, there are two stages. In the first place, you must have a large list of investment projects to choose from. The list must be as diversified as possible, each project in the list being characterised by objective data of various sorts: What the increase in capacity will be if and when we have carried through the investments, what in put effects will be manifest if and when you decide to go through with a specific group of projects and so forth. I have said connection about this in my paper. The essential point to retain in this comment is that when you make this list of projects you must not think that the mere entering of a project in the list is equivalent to saying « I am going to carry this project out ». It is simply putting up an alternative that may or may not be accepted and it is from this big list of investments projects that we have to choose: the list is the basis of the scientific programming of decisions on investments. Similarly we should handle a list of Patterns of Centres.

ALLAIS

I do not see at all what procedure professor FRISCH recommends. If we have to choose investments or to decide what policy must be followed for different regions, the problems to be solved are absolutely different. A planning bureau can choose between different investment projects according to different criteria — that is very simple and clear in principle, but if we have to decide what policy to apply for different regions, the nature of the problem is completely different. In fact, who will decide? The Central Planning Bureau? But if the people of the different regions are not represented in this Central Planning Bureau, what will happen? I will give only one example. In the different Western countries, we have a tariff policy, and the effect of this tariff policy is that in these countries the peripheral regions alone support the burden of this policy. For instance in my own country, Brittany is bearing the cost of protecting the french coal mines, and in Canada, it is in the main the Province of Quebec which supports the Canadian tariff policy of Canada and so on. So, in my opinion, it is impossible to treat these two problems, investment on the one hand and policy for each region on the other in the same way.

FRISCH

I am sorry that my explanation must have been too brief, because Prof. ALLAIS has completely misundertood what I meant to say. When I spoke about *analogy* with investments selection in a big list of projects, I was only referring to the formal construction of what we are going to take as a pattern of centres. There may be many alternative patterns of centres, or there may be many investment projects in our list of projects.

I did not speak at all about how the decision of powers is to be distributed. We have to use this big list of alternatives of centres, and out of those make a choice. The choice is to be studied by mathematical programming.

ALLAIS

Again there is a great difference but I do not think we can discuss the question now.

ISARD

I think this discussion which has been going on is a part of the general problem of deciding upon the criteria to be used in making decisions. The criteria can be expected to, and should differ, from one pattern of decision-making authority to another, from one regional planning group to another, etc. Thus the question of how to make decisions, how to conduct research, what techniques to use must be related to the question of who is to make decisions. They must be related to some system of regional planning authorities, to some spatial pattern of decentralization in decision-making authority. The question of the appropriate spatial pattern of decision-making authority is the concern of my paper. It is a question which must be answered if we are to resolve the issues raised by FRISCH and ALLAIS.

I agree with Professor WOLD that rigorously speaking, complete information is implicitly assumed at certain points in my paper. This is indeed a very stringent assumption. but is required if we are to make any headway on our basic problem.

DORFMAN

Prof. ISARD broaches some really frightening questions, and I am forced to drop from view all but one of them. He mentions that it would be very economical to have a completely decentralized system in which decisions could be made at each point in the light of local conditions. But this is not necessarily so. It is not so if the decision unit in each locality has to base its decision on imperfect guesses about the decisions being made in other localities, which is the normal case if the localities are bound together into a community. An example is the difficulties faced by the independent highway commissions of the United States in trying to establish an intelligible system of highway codes that will not interfere with the flow of traffic.

Some of the experimental results of small group dynamics are pertinent to this issue. If you decentralize the decisions in a group too far, so that each member of a team is required to obtain a lot of information from all the others in order to make his decisions, the efficiency of the whole team is diminished. It is pretty clear that the relationship between efficiency and centralization is not monotonic; you run into serious inefficiencies at both extremes.

HAAVELMO

It seems to me that part of Prof. ISARD's argument has to do with the old question of whether errors committed by centralized decisions will be bigger than the errors of decentralized decisions because, allegedly, the errors in the latter case may tend to cancel each other. But the answer is not so simple. It will depend essentially on the kind of correlations that exist between the errors, and this again depends on the network of relations in the economic system.

ISARD

I am glad to have Professor DORFMAN's comments. They indicate to me that I have not been as clear as I should have been on a number of points. First, I meant to suggest that only in some situations and with respect to only some specific functions (e. g. planning on local education) would a highly decentralized system be desirable. In many other cases, such as in transportation planning, a high degree of spatial decentralization of decision-making authority is highly undesirable.

I also would subscribe to the view that in the operation of many groups the relationship between efficiency and centralization is not monotonic. Actually, I use a number of graphs to indicate the nonmonoticity of this relationship; and most empirical investigation would bear out this hypothesis. Again, I want to emphasize the highly exploratory character of my paper, which represents only an initial attempt to dig away at a problem which constantly plagues every regional-national planner — and a problem which has been by and large ignored by social scientists in their analytic frameworks.

THE RATES OF LONG-RUN ECONOMIC GROWTH AND CAPITAL TRANSFER FROM DEVELOPED TO UNDERDEVELOPED AREAS

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1. The Underdeveloped Areas, which hold at least two-thirds of the entire population of the world, produce now only about one-seventh of the world's gross output of goods and services; moreover, their rate of economic growth is at the present time much lower — possibly only half as high — as that of the advanced industrialized countries. That means that the contrast between the richer and the poorer areas tends to increase rather than diminish.

A rise in the rate of growth of the Underdeveloped Areas would demand an increased volume of productive investment. The additional capital could be created through stepped up internal savings, or it might be obtained from abroad, that is, transferred in the form of aid, foreign loans, or direct private investment from the Developed countries.

How much additional investment would the Underdeveloped parts of the world have to absorb if they were to raise their average growth rate, over the next ten year period, up to the average growth rate of the economically advanced industrial countries? If capital transferred from Developed to Underdeveloped Areas were to constitute the principal source of such additional investment, how large would this transfer have to be?

The simple dynamic system presented below describes in crude aggregative terms the relationships of the magnitude of the capital transfer from Developed to Underdeveloped Areas, and of the levels of saving and investment in both groups of countries to their respective rates of growth. It was designed so as to require not more factual statistical iuformation than is actually available. The over all capital-output and saving ratios of the more and the less advanced countries as well as the proportion (but not of course the absolute amount) of the Gross National Product of the Developed Areas transferred to the Underdeveloped countries are assumed to be constant over the ten year period over which we project their future growth.

Since aggregative capital-output ratios (capital coefficients) and saving ratios can be estimated — particularly for the Underdeveloped Areas — only within a rather wide margin of error and because our expressed purpose is to assess the possible effect of changes in the amount of outside capital received by Underdeveloped Areas on their rate of growth, not one, but many alternative projections were made, all computed from the same general formula, but each based on different hypothetical combinations of the magnitudes of the structural parameters enumerated above.

2. The following set of aggregative variables is used to describe the state of the two groups of economics — Developed and Underdeveloped — at any particular point of time, t:

VARIABLES	Developed Areas	Underdeveloped Areas
Gross National Product (domestically produced) . .	$Y_1(t)$	$Y_2(t)$
Productive investment (total)	$I_1(t)$	$I_2(t)$
Capital transfer from Developed to Underdeveloped Areas	$H(t)$	
Growth rate of the domestically produced Gross National Product, $\frac{\dot{Y}(t)}{Y(t)}$	$r_1(t)$	$r_2(t)$

The value of these seven variables given (that is, observed or assigned) for the year 1959 constitutes the empirical basis of a series of alternative projections of the economic growth of both groups of countries over the ten year period ending at 1969.

DEVELOPED AREAS

The following theoretical relationships are used to derive, and to solve, the equations describing the growth of the Developed Areas:

Saving Function:

$$I_1(t) = i_1 Y_1(t) \tag{1}$$

i represents the fraction of the GNP allocated to investment.

Acceleration Relationships:

$$\dot{Y}_1(t) = \frac{I_1(t)}{b_1} \tag{2}$$

b_1 is the capital coefficient (capital-output ratio) describing the amount of capital required per additional unit of annual GNP.

Growth Rate equation, obtained from (1) and (2):

$$\dot{Y}_1(t) - \frac{i_1}{b_1} Y_1(t) = 0 \tag{3}$$

Exponential Growth Function, obtained by solving (3):

$$Y_1(t) = Y_1(0)\, e^{\lambda_1 t}, \qquad \lambda_1 = \frac{i_1}{b_1} \tag{4}$$

where $Y_1(0)$ represents the level of the GNP in the base year 0 and λ_1 its growth rate, which remains constant as long as i_1 and b_1 are fixed.

The amount transferred from the Developed to the Underdeveloped Areas is assumed to constitute a fixed fraction, h, of the GNP of the capital-exporting countries. Thus, the following Transfer relationship, which is derived from equations (4) above, implies that $H(t)$, the amount transferred, will grow exponentially at the same rate as the Developed Areas' GNP:

Transfer relationship:

$$H(t) = hY_1(t) = hY_1(0)e^{\lambda_1 t} \tag{5}$$

Underdeveloped Areas

The productive investment in the Underdeveloped Areas is being supported from two sources: The saved fraction, i_2, of their own Gross National Product, $Y_2(t)$, and the capital-imports, $H(t)$:

Investment Function:

$$I_2(t) = i_2 Y_2(t) + H(t) = i_2 Y_2(t) + h Y_1(o) e^{\lambda_1 t} \tag{6}$$

Acceleration Relationship:

$$\dot{Y}_2(t) = \frac{I_2(t)}{b_2} \tag{7}$$

b_2 is the capital coefficient describing the amount of capital required per additional unit of annual GNP.

Growth rate equation derived from (6) and (7):

$$\dot{Y}_2(t) - \frac{i_2}{b_2} Y_2(t) - \frac{h}{b_2} Y_1(o) e^{\lambda_1 t} = o \qquad , if \ \frac{i_2}{b_2} \neq \lambda_1 = \frac{i_1}{b_1} \tag{8}$$

Growth Function, obtained by solving the differential equation (8):

$$Y_2(t) = \left[Y_2(o) + \frac{H(o)}{b_2(\lambda_1 - \lambda_2)} \right] e^{\lambda_2 t} - \frac{H(o)}{b_2(\lambda_1 - \lambda_2)} e^{\lambda_1 t} \quad , \lambda_2 = \frac{i_2}{b_2} \tag{9}$$

To verify the last equation one can substitute it and its derivative in (8); the expression on the left hand side will identically equal zero.

The growth of the Underdeveloped Areas turns out to be described by a combination of two exponential terms. The first reflects the effects of internal savings, the second the contribution of investment financed through capital imports. Accordingly the growth rate, λ_2, of the first component depends on the magnitude of the domestic savings and capital-output ratios while the second term grown at the same rate as the GNP of the Developed Areas ([1]).

Equations (4) and (9) permit us to project forward the growth of both groups of countries provided their base year levels of their respective GNP's, Savings, Growth Rates, as well as the initial magnitude of the interregional capital transfer is given. The corresponding values of the constants entering into the two growth functions can be computed from the following formulae:

$$b_1 = \frac{I_1(o)}{\dot{Y}_1(o)} = \frac{I_1(o)}{Y_1(o)\, r_1(o)} \qquad b_2 = \frac{I_2(o)}{\dot{Y}_2(o)} = \frac{I_2(o)}{Y_2(o)\, r_2(o)}$$

(10)

$$\lambda_1 = \frac{i_1}{b_1} = \frac{\dot{Y}_1(o)}{Y_1(o)} = r_1(o) \qquad \lambda_2 = \frac{[I_2(o) - H(o)]\, \dot{Y}_2(o)}{Y_2(o)\, I_2(o)} = \left[1 - \frac{H(o)}{I_2(o)}\right] r_2(o)$$

These relations are obtained by inserting the given base year values of the variables in the appropriate Investment Functions, Accelerations Relationships and Growth Functions.

([1]) If the ratio of the saving to the capital coefficient in the Developed and Underdeveloped regions happens to be equal, the solution of the differential equation (8) is reduced to:

(9-a) $$Y_2(t) = Y_2(o)\, e^{\lambda t} + \frac{H(o)}{b_2}\, t e^{\lambda t} \qquad \lambda = \frac{i_1}{b_1} = \frac{i_2}{b_1}$$

that is, the growth rates of both groups of countries would in this case be equal.

3. The base year values of the variables used for projection, presented in the following tables, are summarized below. The base year is 1959.

	Developed Areas	Underdeveloped Areas
Gross National Product (domestic)	\$2,105 billion	\$195 billion
Productive Investment (total)	\$228 billion	\$15 - \$22 billion
Capital Transfer from Developed to Underdeveloped Areas	\$4 billion	
Growth rate of the domestically produced Gross National Product	4%-6%	2% - 3%

The Developed Areas comprises Western Europe (excluding Spain, Portugal, Greece and Turkey) United States, Canada, Japan, Soviet Russia and other socialist countries; the Underdeveloped Areas comprises all other countries. Since these estimates, compiled from United Nations and other statistical sources, are supposed to cover all countries, they obviously are subject to a very substantial margin of error. The estimates of the annual rate of gross productive investment in the Underdeveloped Areas, which is particularly uncertain, is presented not as a single figure but in terms of two figures — a high and a low — which does not mean of course that the true magnitude still might not lie outside of that range.

For similar reasons the base year estimate of the long-run growth rates of the GNP's is also presented for each area in the form of two percentage figures — a higher and a lower one.

5. The calculations, the results of which are summarized in Tables I to VI, show how the future economic growth of

the Developed and Underdeveloped Areas might be affected by changes in the fraction of the Gross National Product of the first transferred for investment purposes into the second area and also changes in the distribution of the GNP — in both groups of countries — between current consumption and productive investment.

The growth functions (4) and (9) are used to project the GNP of the Developed and the Underdeveloped Areas over a ten year period, 1959-1969.

The structural coefficients entering into these equations depend — see (10) — on the base year magnitude of the GNP's, of the productive investments in both Areas and also on their respective long-run growth rates in the base year.

Each table is based on a different combination of the estimated 1959 growth rates of both countries and of the estimated amount of domestically financed investment absorbed by the Underdeveloped Areas in that year. The estimates of the 1959 GNP's of both Areas and of the amount of gross investment absorbed in that year by the Developed countries remain the same through all the computation.

The first column of figures in each table presents one particular estimate of the base year state of botb groups of countries and also the levels of their respective GNP's « ten years later », projected from 1959 to 1969. This projection is made on the assumption that the domestic saving ratios in both Areas retain their base year magnitudes and that the economically more advanced countries continue, throughout the ten year period in question, to transfer to the less advanced the same percentage (*h*) of their annual GNP as they did in 1959.

The three other columns show how hypothetically postulated changes in original allocation of the GNP's of both areas — if introduced in the base year and then maintained over the ten years covered by our projections — would have affected the levels of their respective GNP's « ten years later », i.e., in 1969. The corresponding average annual growth rates over the period 1959-1969 are entered below.

The allocations of the GNP of the Developed Areas is described by the constants i_1 and h, i.e., the percentage of the GNP invested domestically and the fraction of that Product transferred to the Underdeveloped Areas and invested there. In the Underdeveloped Areas it is described simply by i_2 — the proportion of it saved and invested; the rest of the investment being equal to the amount received from the Developed countries.

In the second column of Table I, the proportion of the GNP of the Developed Areas transferred to the Underdeveloped Areas is assumed to remain — over the ten year period covered by the projection — the same (0.3%) as it was in 1959. However, the domestic investment ratio is assumed to have been raised in the Developed countries from 18.9% to 19.9% and in the Underdeveloped countries from 9.2% to 10.8%. As a result of that the average growth rate of the two Areas — over the ten year period, 1959-1969 — would go up, respectively from 5% to 5.3% and from 3.2% to 3.5%.

In the third column the transfer coefficient, h, is stepped up to 1.2% and in the fourth column to 2.1%. That implies a transfer of $15 billion and respectively $25 billion in 1959, and of correspondingly larger amounts with the growth of the GNP of the Developed Areas in subsequent years.

Comparing the projected average annual growth rates shown in columns 3 and 4 we find that even with the capital transfer stepped up to $15 billion already in 1959, the Underdeveloped countries would still continue to grow slower than the Developed; with $25 billion they would begin to catch up. The « break even point » at which the two growth rates become equal would be reached with a base year capital transfer of somewhere between $15 and $25 — probably around $20 — billion.

The other five tables are organized on the same pattern as the first. The difference between any two tables lies in the assessment of the actual position of the two areas in the base year 1959.

Thus, so far as the original 1959 growth rates are concerned the projections shown on Table II are based on the same 5% and respectively 3% figure as the projections presented on Table I. The actual domestically financed 1959 investment of the Underdeveloped countries is estimated in Table I at $18 billion, while in Table II at only $11 billion. Accordingly, the implicit estimate of the capital-output ratio, b_2, falls from 3.76 on Table I to 2.56 on Table II.

With any given absolute amount of investment, the lower is the capital-output ratio, the higher must be the rate of growth. This explains why in the third projection on Table II with a base year transfer of $15 billion, the Underdeveloped countries attained — over the ten year period, 1959-1969 — an average annual growth rate of 5.4%, which is higher than the corresponding growth rate of the Developed Areas, shown to be 5.2%.

A comparison of the implicit capital-output ratios of the two areas throws light on the plausibility of some of the base year estimates from which the different sets of projections have been derived. For example, in Table VI, $b_1 = 3.15$ and $b_2 = 5.64$. It seems to be quite unlikely that the average capital intensity of production would be so much higher in the less advanced than in the more advanced areas. A combination of a high growth rate for the Developed, with a low growth rate and a high investment figure for the less developed Areas must be rejected as implausible. This throws considerable doubt also on the validity of all the four alternative projections of future economic growth presented on Table VI.

For reasons similar to those described above, or for some other reasons, a critical examination of the alternative factual assessments of the state in which both groups of countries actually found themselves in 1959 might lead some experts to reject, out-of-hand, some other of the 24 different projections presented in these tables.

The examination of all the alternative projections of the prospective growth of the two groups of countries over the ten year period, 1959-1969, enables us in any case to answer the two questions posed in the opening paragraph of this paper. Whichever set of factual assumptions concerning the base year situation one chooses to accept, one reaches the same conclusion that the « break even point » between the rates of the economic expansion of the two groups of countries could not be reached before the Underdeveloped Areas would raise their average annual growth rate to about five percent. To accelerate their present much slower pace, they would have to double the actual 1959 rate of investment in the very first year and then raise it progressively from year to year.

To make this possible the annual transfer of (productively invested) capital from Developed to Underdeveloped Areas would have to increase from $4 to around $15 or even $20 billion in the first year and then go up annually reaching the level of between $28 and $35 billion in the tenth year.

An assessment of the feasibility of such an ambitious investment program for the Underdeveloped countries supported by a massive capital inflow from the Developed countries lies outside the scope of this paper.

The factual conclusion drawn from the numerical results of our computations has to be accepted, or rejected, in the light of the plausibility of the analytical approach and the reliability of the factual information on which they are based. Being fully conscious of the uncertain and even controversial nature of the statistical estimates, which have to be used in such aggregative analysis, I purposefully presented not one or two but a very large number of alternative projections reflecting a wide range of possible initial conditions. The simple analytical system developed for that purpose is constructed in such a way that with a minimum of computational efforts the spectrum of alternative projections can be expanded furtber through insertion in the appropriate computational formula of still other, different figures purporting to give a more correct assessment of the base year situation. Thus, for example, the estimate of the amount of capital transferred in the year 1959 from the Developed to Underdeveloped Areas might possibly be increased from $4 to, say, $5 or even $6 billion. Projections based on such revised descriptions of the base year situation would incidentally show that a larger amount of additional investment in Underdeveloped Areas and a correspondingly greater increase in the level of capital imports from the Developed Areas, than those that were mentioned above, would be required to bring about an equilization in the growth rates of these two groups of countries.

TABLE I — *Economic growth and capital transfer from developed to underdeveloped areas*

	Original Allocation	Alternative Allocations		
	1	2	3	4
DEVELOPED COUNTRIES (growth rate in base year: 5%, implicit $b_1 = 3.78$)				
Base Year (1959)				
Gross National Product	$1205.0 bil.	$1205.0 bil.	$1205.0 bil.	$1205.0 bil.
Capital transfers to underdeveloped areas	$4.0 bil.	$4.0 bil.	$15.0 bil.	$25.0 bil.
as per cent of national product (h)	0.3%	0.3%	1.2%	2.1%
Domestic investment	$228.0 bil.	$240.0 bil.	$237.6 bil.	$235.6 bil.
as per cent of national product (i_1)	18.9%	19.9%	19.8%	19.5%
Total consumption	$973.0 bil.	$961.0 bil.	$952.4 bil.	$944.4 bil.
as per cent of national product	80.8%	79.8%	79.0%	78.4%
Ten Years Later (1969)				
Gross National Product	$1986.7 bil.	$2039.0 bil.	$2029.2 bil.	$2019.9 bil.
Average annual growth rate over the ten year period	5.0%	5.3%	5.2%	5.2%
UNDERDEVELOPED COUNTRIES (growth rate in base year: 3%, implicit $b_2 = 3.76$)				
Base Year (1959)				
Gross National Product	$195.0 bil.	$195.0 bil.	$195.0 bil.	$195.0 bil.
Domestically financed investment	$18.0 bil.	$21.0 bil.	$21.0 bil.	$21.0 bil.
as per cent of national product (i_2)	9.2%	10.8%	10.8%	10.8%
Total investment	$22.0 bil.	$25.0 bil.	$36.0 bil.	$46.0 bil.
Total consumption	$177.0 bil.	$174.0 bil.	$174.0 bil.	$174.0 bil.
as per cent of national product	90.8%	89.2%	89.2%	89.2%
Ten Years Later (1969)				
Gross National Product	$264.7 bil.	$275.7 bil.	$319.6 bil.	$358.3 bil.
Average annual growth rate over the ten year period	3.2%	3.5%	4.9%	6.1%

TABLE II — *Economic growth and capital transfer from developed to underdeveloped areas*

	Original Allocation	Alternative Allocations		
	1	2	3	4
DEVELOPED COUNTRIES (growth rate in base year: 5%, implicit $b_1 = 3.78$)				
Base Year (1959)				
Gross National Product	$1205.0 bil.	$1205.0 bil.	$1205.0 bil.	$1205.0 bil.
Capital transfers to underdeveloped areas	$4.0 bil.	$4.0 bil.	$15.0 bil.	$25.0 bil.
as per cent of national product (h)	0.3%	0.3%	1.2%	2.1%
Domestic investment	$228.0 bil.	$238.4 bil.	$236.1 bil.	$234.0 bil.
as per cent of national product (i_1)	18.9%	19.8%	19.6%	19.4%
Total consumption	$973.0 bil.	$962.6 bil.	$953.9 bil.	$946.0 bil.
as per cent of national product	80.8%	79.9%	79.2%	78.5%
Ten Years Later (1969)				
Gross National Product	$1988.3 bil.	$2036.4 bil.	$2026.8 bil.	$2018.4 bil.
Average annual growth rate over the ten year period	5.0%	5.3%	5.2%	5.1%
UNDERDEVELOPED COUNTRIES (growth rate in base year: 3%, implicit $b_2 = 2.56$)				
Base Year (1959)				
Gross National Product	$195.0 bil.	$195.0 bil.	$195.0 bil.	$195.0 bil.
Domestically financed investment	$11.0 bil.	$11.6 bil.	$11.6 bil.	$11.6 bil.
as per cent of national product (i_2)	5.6%	5.9%	5.9%	5.9%
Total investment	$15.0 bil.	$15.6 bil.	$26.6 bil.	$36.6 bil.
Total consumption	$184.0 bil.	$183.4 bil.	$183.4 bil.	$183.4 bil.
as per cent of national product	94.4%	94.1%	94.1%	94.1%
Ten Years Later (1969)				
Gross National Product	$266.6 bil.	$268.4 bil.	$331.3 bil.	$387.8 bil.
Average annual growth rate over the ten year period	3.1%	3.2%	5.4%	7.1%

TABLE III — *Economic growth and capital transfer from developed to underdeveloped areas*

	Original Allocation	Alternative Allocations		
	1	2	3	4
DEVELOPED COUNTRIES (growth rate in base year: 4%, implicit $b_1 = 4.73$)				
Base Year (1959)				
Gross National Product	$1205.0 bil.	$1205.0 bil.	$1205.0 bil.	$1205.0 bil.
Capital transfers to underdeveloped areas	$4.0 bil.	$4.0 bil.	15.0 bil.	$25.0 bil.
as per cent of national product (h)	0.3%	0.3%	1.2%	2.1%
Domestic investment	$228.0 bil.	$239.0 bil.	$236.0 bil.	$234.0 bil.
as per cent of national product (i_1)	18.9%	19.8%	19.6%	19.4%
Total consumption	$973.0 bil.	$962.0 bil.	$954.0 bil.	$946.0 bil.
as per cent of national product	80.8%	79.9%	79.2%	78.5%
Ten Years Later (1969)				
Gross National Product	$1784.0 bil.	$1818.0 bil.	$1818.0 bil.	$1801.0 bil
Average annual growth rate over the ten year period	4.0%	4.2%	4.2%	4.1%
UNDERDEVELOPED COUNTRIES (growth rate in base year: 2%, implicit $b_2 = 3.85$)				
Base Year (1959)				
Gross National Product	$195.0 bil.	$195.0 bil.	$195.0 bil.	$195.0 bil.
Domestically financed investment	$11.0 bil.	$11.6 bil.	$11.6 bil.	$11.6 bil.
as per cent of national product (i_2)	5.6%	5.9%	5.9%	5.9%
Total investment	$15.0 bil.	$15.6 bil.	$26.6 bil.	$36.6 bil
Total consumption	$184.0 bil.	$183.4 bil.	$183.4 bil.	$183.4 bil
as per cent of national product	94.4%	94.1%	94.1%	94.1%
Ten Years Later (1969)				
Gross National Product	$240.0 bil.	$240.0 bil.	$275.0 bil.	$309.0 bil.
Average annual growth rate over the ten year period	2.1%	2.1%	3.5%	4.7%

TABLE IV — *Economic growth and capital transfer from developed to underdeveloped areas*

	Original Allocation	Alternative Allocations		
	1	2	3	4
DEVELOPED COUNTRIES (growth rate in base year: 4%, implicit $b_1 = 4.73$)				
Base Year (1959)				
Gross National Product	$1205.0 bil.	$1205.0 bil.	$1205.0 bil.	$1205.0 bil.
Capital transfers to underdeveloped areas	$4.0 bil.	$4.0 bil.	$15.0 bil.	$25.0 bil.
as per cent of national product (h)	0.3%	0.3%	1.2%	2.1%
Domestic investment	$228.0 bil.	$239.0 bil.	$236.0 bil.	$234.0 bil.
as per cent of national product (i_1)	18.9%	19.8%	19.6%	19.4%
Total consumption	$973.0 bil.	$962.0 bil.	$954.0 bil.	$946.0 bil.
as per cent of national product	80.8%	79.9%	79.2%	78.5%
Ten Years Later (1969)				
Gross National Product	$1784.0 bil.	$1818.0 bil.	$1818.0 bil.	$1801.0 bil.
Average annual growth rate over the ten year period	4.0%	4.2%	4.2%	4.1%
UNDERDEVELOPED COUNTRIES (growth rate in base year: 2%, implicit $b_2 = 5.64$)				
Base Year (1959)				
Gross National Product	$195.0 bil.	$195.0 bil.	$195.0 bil.	$195.0 bil.
Domestically financed investment	$18.0 bil.	$19.0 bil.	$19.0 bil.	$19.0 bil.
as per cent of national product (i_2)	9.2%	9.7%	9.7%	9.7%
Total investment	$22.0 bil.	$23.0 bil.	$34.0 bil.	$44.0 bil.
Total consumption	$177.0 bil.	$176.0 bil.	$176.0 bil.	$176.0 bil.
as per cent of national product	90.8%	90.3%	90.3%	90.3%
Ten Years Later (1969)				
Gross National Product	$238.0 bil.	$242.0 bil.	$267.0 bil.	$291.0 bil.
Average annual growth rate over the ten year period	2.0%	2.2%	3.2%	4.1%

TABLE V — *Economic growth and capital transfer from developed to underdeveloped areas*

	Original Allocation	Alternative Allocations		
	1	2	3	4
DEVELOPED COUNTRIES (growth rate in base year: 6%, implicit $b_1 = 3.15$)				
Base Year (1959)				
Gross National Product	\$1205.0 bil.	\$1205.0 bil.	\$1205.0 bil.	\$1205.0 bil.
Capital transfers to underdeveloped areas	\$4.0 bil.	\$4.0 bil.	\$15.0 bil.	\$25.0 bil.
as per cent of national product (h)	0.3%	0.3%	1.2%	2.1%
Domestic investment	\$228.0 bil.	\$239.0 bil.	\$236.0 bil.	\$234.0 bil.
as per cent of national product (i_1)	18.9%	19.8%	19.6%	19.4%
Total consumption	\$973.0 bil.	\$962.0 bil.	\$954.0 bil.	\$946.0 bil.
as per cent of national product	80.8%	79.9%	79.2%	78.5%
Ten Years Later (1969)				
Gross National Product	\$2158.0 bil.	\$2220.0 bil.	\$2220.0 bil.	\$2199.0 bil.
Average annual growth rate over the ten year period	6.0%	6.3%	6.3%	6.2%
UNDERDEVELOPED COUNTRIES (growth rate in base year: 2%, implicit $b_2 = 3.85$)				
Base Year (1959)				
Gross National Product	\$195.0 bil.	\$195.0 bil.	\$195.0 bil.	\$195.0 bil.
Domestically financed investment	\$11.0 bil.	\$12.0 bil.	\$12.0 bil.	\$12.0 bil.
as per cent of national product (i_2)	5.6%	6.2%	6.2%	6.2%
Total investment	\$15.0 bil.	\$16.0 bil.	\$27.0 bil.	\$37.0 bil.
Total consumption	\$184.0 bil.	\$183.0 bil.	\$183.0 bil.	\$183.0 bil.
as per cent of national product	94.4%	93.8%	93.8%	93.8%
Ten Years Later (1969)				
Gross National Product	\$242.0 bil.	\$242.0 bil.	\$286.0 bil.	\$324.0 bil.
Average annual growth rate over the ten year period	2.2%	2.2%	3.9%	5.2%

TABLE VI — *Economic growth and capital transfer from developed to underdeveloped areas*

	Original Allocation	Alternative Allocations		
	1	2	3	4
DEVELOPED COUNTRIES (growth rate in base year: 6%, implicit $b_1 = 3.15$)				
Base Year (1959)				
Gross National Product	$1205.0 bil.	$1205.0 bil.	$1205.0 bil.	$1205.0 bil.
Capital transfers to underdeveloped areas	$4.0 bil.	$4.0 bil.	$15.0 bil.	$25.0 bil.
as per cent of national product (h)	0.3%	0.3%	1.2%	2.1%
Domestic investment	$228.0 bil	$239.0 bil.	$236.0 bil.	$234.0 bil.
as per cent of national product (i_1)	18.9%	19.8%	19.6%	19.4%
Total consumption	$973.0 bil.	$962.0 bil.	$954.0 bil.	$946.0 bil.
as per cent of national product	80.8%	79.9%	79.2%	78.5%
Ten Years Later (1969)				
Gross National Product	$2158.0 bil.	$2220.0 bil.	$2220.0 bil.	$2199.0 bil.
Average annual growth rate over the ten year period	6.0%	6.3%	6.3%	6.2%
UNDERDEVELOPED COUNTRIES (growth rate in base year: 2%, implicit $b_2 = 5.64$)				
Base Year (1959)				
Gross National Product	$195.0 bil.	$195.0 bil.	$195.0 bil.	$195.0 bil.
Domestically financed investment	$18.0 bil.	$19.0 bil.	$19.0 bil.	$19.0 bil.
as per cent of national product (i_2)	9.2%	9.7%	9.7%	9.7%
Total investment	$22.0 bil.	$23.0 bil.	$34.0 bil.	$44.0 bil.
Total consumption	$177.0 bil.	$176.0 bil.	$176.0 bil.	$176.0 bil.
as per cent of national product	90.8%	90.3%	90.3%	90.3%
Ten Years Later (1969)				
Gross National Product	$240.0 bil.	$242.0 bil.	$272.0 bil.	$297.0 bil.
Average annual growth rate over the ten year period	2.1%	2.2%	3.4%	4.3%

DISCUSSION

ALLAIS

Prof. LEONTIEF's paper is very interesting and the ideas he expressed and his conclusions are very important and stimulating from two points of view: first, theoretical and secondly political. And in view of this importance I find myself obliged to express my disagreement, in a very friendly, but very firm way. This disagreement relates to equation 2, which is founded on the same hypothesis as Prof. HAAVELMO's paper yesterday. This equation 2 has practically underpinned Western policy over the last 15 years. In every meeting of the United Nations, equation 2 has been implicitly accepted and has thus had great importance. I think it is worth while to discuss this equation 2 and the hypothesis on which it is founded in detail. The hypothesis is that the real national income of any one country is proportional to its real capital.

I think this proposition can be accepted neither from a theoretical nor an empirical point of view. We have to distinguish two different points carrefully. The first is: we observe that the capital output ratio is practically constant for a number of countries at different times. That much is certain and I have personally given some data to support this conclusion in my 1961 Econometrica paper. But it is a completely different thing to accept that in every situation there is proportionality between real income and real capital. Perhaps my point of view is quite difficult to understand because it is not very common.

I believe this point is absolutely independent of the model given by my paper, but it is in the framework of this model that my view can be put most clearly. In the model, the conclusion is reached that the capital-output ratio must be practically constant. Nevertheless, the conclusion is reached in the same model that there is a maximum value for real national income, whatever the level of real capital.

May I recall that in my model, in the exponential case,

$$\gamma = \frac{C}{R} = \frac{\Theta_0}{1 + \Theta_0 i} \qquad \text{(relation 251-6)}$$

where γ is the capital output ratio, C the nominal value of capital, R the nominal value of the national income, i the rate of interest and Θ_0 a constant whose order of magnitude is 4. For small values of i, γ is practically a constant.

But we also have in the exponential case

$$\frac{\bar{R}}{\bar{R}_M} = \left[\frac{\Theta_0}{\gamma} e^{1 - \frac{\Theta_0}{\gamma}} \right]^k \qquad \text{(relation 251-15)}$$

where $\bar{R}$ is the real value of the national income, and k the coefficient of homogeneity of the production function. R_{CM} is the maximum value of $\bar{R}$ for $\gamma = \Theta_0$

$\bar{R}$ *has a maximum whatever the value of* $\bar{C}$. Nevertheless γ varies little over the usual range of variation of i.

Here, therefore, is at least one system of consistent hypotheses and mathematical deductions in which there are at the same time two valid propositions. The first is that when the rate of interest is small, the capital output ratio is practically constant.

[13] *Leontief* - pag. 20

At the same time there is a second proposition according to which real national income has a maximum, whatever the level of real capital.

Of course, this model in no way constitutes a proof, but at least it supports the fact that the two propositions are absolutely different. It also shows that it is impossible to derive a general property of the production function, from the fact that the capital output ratio is practically constant, according to which real national income would be proportional to real capital. Thus from a theoretical point of view it is impossible to deduce proportionality between real national income and real capital from the fact that in certain circumstances we can observe an apparent constancy of the capital output ratio.

From an empirical point of view the empirical researches of Douglas and his followers have shown that we can write at least as a first approximation

$$\overline{R} = KL^{\alpha}\overline{C}^{\beta}$$

where $\overline{R}$ is real national income, L labor, C real capital, and K, α, β constants.

The Douglas results have been very much discussed but there is one point which has never been contested. This is that the order of magnitude of the elasticity

$$\beta = \frac{\overline{C}}{\overline{R}} \frac{d\overline{R}}{d\overline{C}}$$

$$= \gamma \frac{d\overline{R}}{d\overline{C}}$$

is 0.2 instead of 1 as is implicitly assumed in equation 2 of LEONTIEF's paper. In no case it is possible to conclude that there is some proportionality between R and C which would mean β equal to unity.

Now, in the Leontief calculations the hypothesis that real national income tends to be proportional to real national capital plays a fundamental role. If this hypothesis were abandoned, the results would be completely different.

My argument is absolutely the same as yesterday. A given property can appear to be reasonable without being right. I repeat again. From an economic point of view, my model may be right or wrong, this is as may be, but it is mathematically coherent. And while it shows an approximate proportionality between income and capital, nevertheless there is at the same time *a maximum* for real national income whatever the value of real capital.

The political impact of the LEONTIEF paper, were its results correct, would be of very great importance. The conclusion would be that the West should very substantially increase its aid in capital. My convinction is that this increase would have a much lesser effect than that predicted by the LEONTIEF paper because the more important factor is not capital but, in my opinion, technological education.

Certainly, if we accepted a production function such as the Douglas production function, which it seems very reasonable to accept, at least as a first approximation, the result would be that the paper would predict a much lesser effect. Thank you.

LEONTIEF

I think I fully understand Prof. ALLAIS' objection. To meet it let me restate my position in respect to the point raised by him in the following way:

The stock of capital employed in productive process represents a necessary but not a sufficient condition for attaining the level of output corresponding to the fixed capital/output ratio entering in my equation 2.

Accordingly, my computations show essentially the upper limits of the future growth in national income that can be expected to be brought about by various combinations of domestic savings and international capital transfer. If the supply of other factor — such as skilled labor, efficient management and natural resources — could not expand proportionately the actual increase in income would in each case fall short of that, in a certain sense, maximal rate.

I used the capital/output ratio essentially as a limiting factor. Not saying that this amount of capital transfer is sufficient to produce that output, I on the other hand imply that without the indicated rate of capital transfer the corresponding rate of income growth — computed on the basis of my formulae — could hardly be attained. Professor ALLAIS does not seem to disagree with this. So far as the political conclusions — which he expects to be drawn from the results of my analysis — are concerned, he might be right or he might be wrong; it all depends on the additional — political — premises which will have to be introduced before one would be justified to draw them.

ALLAIS

I agree completely with what Professor LEONTIEF has said but I wish to stress two points. First, precisely in this particular line, an attempt is made to calculate some limits of the capital effect, and this attempt is very interesting indeed. But if this is done, b should be replaced in equation 2 by b/β where β is equal to 0.2 and not 1, if we accept the results of the Douglas researches and those of his followers. My second point is that, so far as underdeveloped countries are concerned, we must be very cautious in extrapolating data which are valid for the West. This, I think, is the error that Douglas himself has committed in assuming that his relation could represent the production function over a very large range. What he has shown is that there is a correlation over time and in space between three quantities: production, labor and capital. But it is impossible to derive the conclusion that this function is valid over the whole range of variation: it holds for the very restricted domain of data relating to some Western developed countries. Thus, I could accept the Leontief calculation, but only on two conditions. The first one would be the replacement of b by b/β with β equal to 0.2 in equation 2. The second would be to say that this is a very tentative calculation and that it is by no means certain that the results would remain as valid as they may be for the West.

HAAVELMO

This exposé was very clear, there is little possibility of misunderstanding. I am actually asking a question about an alternative hypothesis, and whether the author has considered it. If you would kindly turn to page 5, equation (6). I just wonder whether LEONTIEF has considered the possibilities that equation (6) might have the alternative form $I_2(t) = i_2[Y_2(t) + H(t)]$. That is to say, that investment in the region is a fraction of regional income including what they get from abroad. I think we see the possible implications. It means, for example, that domestic savings could be negative if H is very large.

LEONTIEF

I have not tried to perform an alternative set of computations based on Professor HAAVELMO's suggestion that the capital inflow into a developing country be treated as a part of its total current income of which a fixed fraction be allocated to productive investment. On this assumption a still larger capital transfer from developed into the less developed countries would be required to narrow down the gap between the growth rates in their respective national incomes.

If I may return with just one more remark to Prof. ALLAIS' comments: I did not make only one computation, I made very many of them covering a wide range of different factual assumptions. Examining the implicit magnitudes of capital coefficients reflecting various estimates of the present rate of income growth and of productive investment in underdeveloped countries, we find them to be in some cases considerably higher, and in others much lower than could be realistically expected. In most instances they appear to be quite plausible. I am inclined to interpret such consistency as representing indirect evidence in support of my conclusions.

FISHER

Professor LEONTIEF has presented a highly interesting paper. I find most attractive the idea of watching economic history with accompanying illustrations. LEONTIEF has in fact produced interesting pictures of what happens when the economy is open to foreign trade and foreign trade either outgoing or ingoing is cut off. One might also consider applying the same technique to the study of the dependence of the economy on particular industries. For example, one might open the system to the steel industry and see what would happen if deliveries from or to that industry were cut off. The pictures one would get from applying LEONTIEF's technique to such analyses would provide interesting insights into dependence of the economy on particular industries or sectors.

ALLAIS

Prof. LEONTIEF has very rightly stressed that he has made calculations with different values of the coefficients b_2 and b_1. But my point is that it is not sufficient to consider values of the capital output ratios b_1 and b_2. It is also necessary to take into account the coefficient β which is of the order of 0.2. Then the b would be replaced by values five times greater, and the range which has been considered is not sufficient. With correct values of the b the results would be much lower.

MAHALANOBIS

I think this is a very important question and I should like to congratulate Professor LEONTIEF for the presentation of his paper. But I find some difficulty because he has not stated explicitly the population figures as my observations would depend possibly on the factual basis of the partition into the developed and the underdeveloped. Using a notional figure of three thousand million as the total population of the world, I am asking whether China has been included with six hundred million, again as a notional figure?

Secondly, assuming that some rates of growth of population have been used, whether such growth was taken into account in estimating the effect on income?

The conclusion is that the rate will be such and such, and 20 or 25 billion (??) dollars of aid would be required per year. What is the implication? Does the aid have to continue almost indefinitely? What I am really trying to do is to bring in the well-known concept of assisted take-off. Now supposing that there would be an assisted take-off in 15 or 20 years, what would be the total effort required? This seems to me to be a realistic and important question.

THEIL

I am much interested in Professor LEONTIEF's paper, partly because it is so close to an analysis which I carried out myself a number of years ago (published in English in the *International Economic Review*). I was much impressed by the large differences in the per capita incomes of the various countries. Accordingly, my criterion was to reduce the variance of the logarithmic per capita income distribution by a certain amount, year after year. The criterion was pursued by means of a system of aid from developed countries to underdeveloped countries along the lines of a model which is close to that of Professor LEONTIEF's. The result was embarrassing. The criterion implied that one country had to pay, viz., the United States, and all others (including Canada, Switzerland, Sweden and Australia) received aid. This unpractical result induced me to change the criterion to that of aid to countries whose present per capita incomes are below a certain level in such a way that they will reach a given « marginal » level within a given period of time. This marginal level was not considered to be constant over time. An exponentially increasing function was preferred in view of the consideration that our present standards for what is marginally acceptable will probably be below the standards that will prevail after 25 years.

ALLAIS (1)

If we take as a provisional hypothesis that according to the Cobb Douglas formulation

$$\bar{Y} = K L^{\alpha} \bar{C}^{\beta}$$

where $\bar{Y}$ is real national income, L is labor and C real capital, then

$$\frac{d\bar{Y}}{\bar{Y}} = \beta \frac{d\bar{C}}{\bar{C}}$$

(1) Comments presented in plenary session. The preceeding comments were presented in the separate group.

and

$$\frac{d\bar{Y}}{\bar{Y}} = \frac{\beta}{b} I$$

where

$$b = \frac{\bar{C}}{\bar{Y}}$$

according to Professor LEONTIEF's definition of page 3.

Then equation (2) must be written

$$(2') \qquad \frac{dY_1}{dt} = \frac{\beta}{b_1} I_1(t)$$

instead of

$$(2) \qquad \frac{dY_1}{dt} = \frac{1}{b_1} I_1(t)$$

Now there is no doubt that the order of magnitude of β is 0.2 and not 1 as in equation 2 of the paper. If 0.2 instead of 1 is taken for β the final results of the paper change very greatly.

In fact the range which Prof. LEONTIEF has considered for b is as follows: For b_1 3.15 to 4.73 and b_2 2.5 to 5.64, the range is about 1 to 2. But if we take the coefficient β equal to 0.2 into account the range which should be considered is five times greater since we must consider b/β that is 5 b instead of b.

We must thus replace the coefficient b by the product $b/\beta = 5\ b$ in the calculations which follow, and this changes the final results completely.

[13] *Leontief* - pag. 28

I think this is very important because, since, especially for India in the last 25 years, western aid policy for underdeveloped countries has been based on the hypothesis that real national income is proportional to real capital. Nobody has questioned this and we have spent many billions of dollars in a way which may have been less efficient than for examples, educating the people. In my opinion, if we consider all the variables which significantly influence development, I would say that capital is less important than other factors, and if this thesis were to be accepted, we should spend the greatest part of our help in other directions.

LEONTIEF

As author of a paper I would like to comment at length on the interesting questions raised in the course of the subsequent discussion; as Chairman, I have the responsibility for adjourning this meeting within a few minutes. Thus, I will make these closing remarks very brief.

In answering Professor MAHALANOBIS' query, I must explain that the United Nations figures which I used as a basis for my calculations include Continental China as they do Soviet Russia among the so-called centrally planned economies which I had to treat as belonging to the group of industrialized countries.

All my computations were conducted on a total not-per capita basis. Translated into per capita terms, the discrepancy between the present growth rates of the underdeveloped and the industrialized parts of the world would of course appear to be still greater and the volume of the capital transfer required to eliminate, or at least to reduce this difference within the next ten years — still larger.

So far as the concept of a take-off period is concerned, I suppose it might be applied to the entire ten-year time span covered by my hypothetical computation.

I fully agree with Professor ALLAIS' emphasis on the importance of educational expenditure as a means of accelerating economic growth. It has often been suggested that such expenditure should be regarded as a part of productive investment. As a quantitative measure of the total stock of such investment at any given point of time one should possibly use the integral, i.e., the cumulated amount of educational expenditure incurred annually over the preceeding say thirty or fifty years.

THE SOCIAL TRANSFORMATION FOR NATIONAL DEVELOPMENT

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1. Introduction

1.1. The problem of improving the material and cultural conditions of the poorer countries of the world has been engaging serious attention during the post-war period. The desire for political independence is rapidly increasing and will continue to grow in the countries still under colonial rule. Also, more and more countries are becoming and will become politically independent. With the gaining of independence, it is being increasingly realized that political freedom is necessary, but is not enough. In most of the underdeveloped areas, attention is being given increasingly to economic development to improve the level of living, by increasing the flow of goods and services and by expanding facilities for cultural amenities. It is also being increasingly appreciated that rapid economic growth can be brought about only by an increasing accumulation of capital to supply modern tools and machinery for new and expanding productive activities which would, in time, solve problems of unemployment or under-employment, and would also continually improve the level of living. Such accumulation of capital would call for increasing domestic savings, and the utilization of such savings for productive purposes. The choice of productive activities (that is, of investments) must also be such as to secure the best possible rate of economic growth over a time horizon of a generation or more.

1.2. How to bring this about? This is where the question of social transformation becomes relevant. Some broad general principles may perhaps be stated with confidence.

2. The Structural Transformation

2.1. It is necessary to give opportunities for participation in productive activities to the largest number of people, and as soon as possible, to all such people as are capable of undertaking such work, and also to utilize available resources in the most effective way for the benefit of the nation as a whole. To create fullest opportunities for rapid growth, it is necessary to remove all barriers to the effective utilization of productive forces, by the people, for the benefit of all the people of the country.

2.2. There are many facets to the problem, some of which are general and some peculiar to particular countries. It is not possible to arrange them in any clear order of priority. In fact the heart of the problem is to make changes in all necessary directions at the same time, in a balanced way, so as to bring about the structural transformation as quickly as possible.

2.3. The transformation of the social structure cannot be an entirely internal process. Outside influences have been and will continue to be at work. Colonial rule and economic exploitation of the underdeveloped countries have themselves given rise to reactions promoting the desire for political independence and for improvement in the level of living in the underdeveloped areas.

2.4. A new factor, of conscious international cooperation in improving the social, political, and economic conditions of the underdeveloped countries, has also emerged during the last ten or fifteen years through a quickening of the world conscience on humanitarian grounds and also in the enlightened self-interest of the more advanced countries. Isolation is no longer possible, physically, psychologically or organizationally. The influence of information, ideas, advice and aid from outside would be an increasingly important factor.

2.5. A structural transformation of the whole society is, however, indispensable to make conditions fit for rapid economic growth. Without such transformation, any amount of help from outside would be ineffective. The experience of many countries during the post-war period would corroborate this.

3. The Scientific Revolution

3.1. It is also necessary to develop the outlook of science and the experimental attitude of mind in order to acquire knowledge of natural and social forces and to invent new techniques for initiating material and social changes. This is the only way in which decisions can be made increasingly in a rational manner, in accordance with principles of objective or scientific validity based on relevant data and correct reasoning, instead of on the sanction of authority based on status and power or custom and conventional or revealed rule and laws. This may be called the scientific revolution.

3.2. The need of what I have called « the scientific revolution » is recognized, but has not received sufficient attention. I have considered some aspects of this problem in an attached note on « The Scientific Base of Economic Development ».

4. Modernization of Society

4.1. The social transformation and the scientific revolution are both necessary. These are but two aspects of modernization which can be distinguished but not separated. The social transformation and the scientific revolution in combination leads to modernization. The task of international cooperation is to promote and help, in every possible way and in a peaceful manner, the modernization of the underdeveloped countries.

4.2. *Urgency of the task*: The scientific and industrial revolution took place in West Europe and North America roughly over a period of three or four hundred years. It is not possible to wait for such a long time for the underdeveloped countries to attain a reasonable level of living. The historical process of transformation must proceed five or ten times faster. Such speeding up of the process of transformation has always been a characteristic feature of biological evolution, and can be achieved.

4.3. *Different phases of the transformation*: Some of the newly independent countries are large, some are of medium size, and some are extremely small in area, or in natural resources or in population. They would have widely differing needs. The particular form and contents and components of each step of modernization would depend on the special conditions of each country and the stage of development reached by it, and would therefore, vary from one country to another or from one region to another of the same country and also, over a period of time, in the same country or in the same region.

4.4. *International cooperation*: The most significant fact of the present age is the rapidly expanding contacts between different countries of the world. This tendency is bound to become stronger in future, increasing the scope of international affairs in every direction. At the same time, what George Washington had said about « no country being able to go beyond its own self-interest in international affairs », would continue to remain valid. The real need is, therefore, to discover new areas of mutual self-interest, and to expand spheres of common interest on both bi-lateral and multi-lateral basis to the fullest extent.

4.5. It is also clear that even the most advanced countries still have unlimited scope for both social and scientific progress. For all countries, large or small, advanced or underdeveloped, international cooperation is necessary and beneficial. The smaller and the less-developed a country, the greater however will be the need and importance of such cooperation.

5. Problems of Internal Reforms

5.1. There is general agreement about some of the most important contents or elements or aspects of the social transformation, such as: — land reform; removal of social, economic and political barriers; mass education and technical training; increasing equality of opportunities; the possibility of a labourer or an initiator securing the fruits of his labour; or the need of medical and health services and cultural amenities etc. There is much in common in respect of such components or aspects of social transformation in the case of all under-developed countries, with, however, the need of adaptations to suit the special conditions of each individual country. Some of these components or aspects are briefly considered below.

5.2. *Land reform*: Historically, land reform has been a most important factor in the economic development of all advanced or rapidly developing countries. Agriculture and industry must advance at the same time. It is, however, generally agreed that an agricultural surplus (or, the surplus from extractives) is essential for industrial development. Changes in land tenure and legislation would, therefore, be one of the requirements of the highest priority in most, if not all, underdeveloped regions.

5.3. The aim must be to secure the fruits of his labour to the cultivator so that he has the incentive to improve the land and to introduce more advanced technological methods. Tenancy law should protect the tenant against eviction so long as he is using the land efficiently, and to secure to him the right of fair compensation upon termination of the lease for all unexhausted improvements made by him. It is also necessary to eliminate the unproductive consumption of the surplus from land by intermediaries and landlords, who have no productive functions, by abolishing their rights.

5.4. The question of economy of scale of production may, admittedly, introduce difficulties. The breaking up of large farms may lead to a reduction of the surplus; however, the beneficial effects of greater equality of income and wealth may compensate for the other loss. Also, in countries where there are too many cultivators, often with scattered plots, further breaking up of the holdings may easily have adverse effects on the efficiency of production. In such a situation it may be necessary to promote consolidation of holdings either voluntarily through cooperative, or by legislation, or both. Redistribution of land has limits and is a complicated question. It is wise to recognize that steps taken at one stage may have to be reversed at a later stage. Appropriate measures must be devised to suit the needs of each country at any particular stage of development. The basic aim would always remain the same, namely, to increase the agricultural surplus, and to use it for productive purposes, as effectively as possible, in speeding up the growth of the economy as a whole.

5.5 *Removal of social, economic, legal and political barriers*: The underdeveloped countries have the very difficult task of achieving a far faster rate of growth than had been achieved by the most advanced countries during and after the industrial revolution. It is indispensable that every one in the working age-group should be fully utilized to increase the national product. It is necessary, therefore, to remove all social, economic, legal and political barriers which prevent individuals, or groups and sections of individuals, to become fully productive. Conditions are worst in a country stratified by caste, colour, creed or language, and where whole sections of people are sometimes deprived of opportunities by custom, law, or social and political pressures by ruling groups. Removing all such barriers is an essential condition for rapid growth.

5.6. *Equality of opportunities and vertical mobility*: Removal of social and other barriers, in principle, is necessary but not sufficient. It is essential to help every one to make himself fit for the highest type of productive work of which he is capable. Opportunities for education and training and for productive work must be made as widely available as possible. Great inequalities of wealth and income often lead to denial of opportunities to the poorer people, and, unless removed, give rise to a sense of frustration among the under-privileged and hamper the growth of national solidarity. Sufficiently rapid economic progress would be difficult or impossible in societies in which there is lack of vertical mobility and where small sections try to preserve their privileges based on heredity, custom or law without any relation to their productive contributions.

5.7. *Horizontal mobility*: The social system may also hamper the utilization of resources because customs or caste restrictions prevent labour from moving into new occupations, or labour is tied to the soil, or land may be concentrated in the hands of small sections of the people who are unwilling to divert it for more productive use for reasons of social or political prestige. A small number of producers even in underdeveloped countries may sometimes band together to prevent the free entry of others or the introduction of new techniques. All such restrictions must be removed to increase the horizontal mobility of resources.

5.8. *Possibility of securing fruits of labour and enterprise*: The elimination of concentration of social, economic or political privileges in the hands of small sections of the people would promote both vertical and horizontal mobility, and make it possible for every one to secure a fair share of the fruits of his labour and enterprise. This is one of the most important consequences of the social transformation and is particularly helpful in promoting rapid economic growth. Appropriate legal and institutional changes must be made to achieve this.

6. National Integration

6.1. *Sectional interests and barriers*: A characteristic feature of underdevelopment is the segmentation of the country into innumerable regions, castes, tribes, languages, religious communities, occupational and other groups which focus attention on the welfare of small sections of the people without any awareness of the best interests of the country as a whole. It has to be recognized that rapid progress is impossible without painful adjustments and damage to sections of the people whose interests are based on special privileges or old techniques; and that old beliefs, customs, and social institutions have to be discarded, and all barriers of caste, customs, creed, colour, language and sectional interests must be ruthlessly eliminated. The greater the prevalence of such social barriers in the country, the greater are the sectional rigidities within government administration, and the fiercer are the inter-agency jealousies and fights which continually delay decisions and hamper speedy action.

6. . *Integration*: The removal of social and economic barriers is an indispensable condition for the emergence of the sense of national solidarity without which national development is impossible. It is not possible to isolate the scientific, or the social, or the industrial aspects of the transformation from one another. Advance must be made at the same time on all fronts. This creates difficulties but also has its advantages. Progress in one direction stimulates and promotes progress in another direction. It is the task of leadership to maintain a proper balance between the different aspects and phases of the

process of modernization in its full sense. The aim continually must be to create a society from which social, economic and political privileges have been completely eliminated. To bring about such a transformation would call for wise leadership with a clear appreciation of aims and objective, a rational and experimental attitude of mind with confidence in the outlook of science, and willing to pay the price of much painful adjustments.

7. Danger of Superficial Imitation of Advanced Countries

7.1. Because of the sense of urgency for economic growth which is strengthening everywhere, there is a peculiar danger of adopting, in a superficial way or at too early a stage, methods, and forms and institutions, which are working successfully in the advanced countries. It has to be kept in mind that existing social and political institutions, or high levels and standards of quality or performance, were established in the advanced countries only with the gradual growth of the economy. Such institutions may not be useful at an earlier stage of development, and may even hamper progress. For example, in underdeveloped countries, there is sometimes a tendency to adopt too expensive or too sophisticated schemes of education, care of health, public buildings and construction, wages or salaries of government employees or labour legislation.

7.2. *Education and training*: Mass education to spread literacy both among children and adults has special urgency; here all possible help should be utilised, for example, by using the services, for a small part of the day or the week, of those who are already literate. Because the number involved are very large, the adoption of too high a standard for teacher qualifications, school buildings etc., at the primary level, would make the cost prohibitive. At the secondary stage, more attention would have to be given to the qualification of teachers and other educational aids; but scales of pay or cost of buildings should still be kept in balance with the general level of living of the students and the parents themselves. At the tertiary level, still higher standards would have to be adopted for staff qualifications and there would be need of more expensive teaching aids; but the expenditure must be kept within the limits of what the country can afford. It is at the stage of advanced studies and research that standards should be really high and comparable with the advanced countries; however, as the number of advanced and research workers would be very small in the beginning, this would not involve any large total expenditure.

7.3. The educational system should be viewed as a pyramid; the lower the stage the wider should be the base (that is, the number of persons under instruction) and the lower the scales of expenditure compared to advanced countries, while at the highest stage of advanced studies and research the number involved would be extremely small but scales of expenditure may approximate to those of advanced countries. Adoption, at too early a stage, of standards and scales of expenditure of advanced countries at lower levels would lead to severe restrictions in numbers usually coupled with admission of students on the basis of family income; this must have most undesirable social and psychological consequences. When resources in men, materials and money are inadequate, to increase the number of students in accordance with the pressure on admissions, would necessarily lead to window dressing and a dilution of standards in practice. This can seriously hamper progress; the only remedy is to adopt a system which would be in keeping with basic aims and yet within the means of the country.

7.4. *Medical care and technical services*: A similar situation can arise even more easily in the field of medical care. Adoption of the high level of university education for physicians in the advanced countries as the only standard at an early stage would necessarily mean that most of the people will be deprived of medical services in underdeveloped countries for a very long time. A two-tier approach with a junior cadre of medical personnel with, say, three or four years' training, together with a much smaller number of physicians with university training, would make it possible to spread medical services much wider and much faster. This would be equally true in many other lines of technical work. A two or even a three-tier approach with a higher, a medium, and even a third level of workers who have had a very quick and specialized training, would be not only within the means of the underdeveloped countries but may be even more effective, because, in the still backward conditions of the country, the lower level workers would be much nearer to the general population and would be able to work in closer touch with them. This would be particularly true in agricultural extension and other services which would bring the technical workers into contact with large sections of the population.

7.5. *Government expenditure*: Government expenditure often tends to become unduly large in underdeveloped countries owing to the adoption of the much higher standards of advanced countries. This leads to unnecessarily high scales of wages and salaries for government employees or costly public buildings; which, in its turn, would increase the feeling of separation between government and the people, and hamper national integration.

7.6. *Labour legislation*: As production becomes modernized and factories and enterprises grow in numbers and in size, it would be necessary to develop labour legislation and regulations to ensure that labour secures a fair share of the surplus, and also to ensure working conditions being maintained reasonably safe and healthy. Legislation in imitation of the more advanced countries, at too early a stage, may, however, lead to increasing inefficiency of performance, especially, in countries with surplus labour, and may increase costs of production so much as to have serious adverse effects on exports. The most important thing is to establish a definite link between remuneration and output in the case of all types of work of which the volume and quality can be estimated even roughly. It is necessary to recognise that trade union movements can gain in real strength only on the basis of increasing productivity.

8. National Leadership

8.1. The transfer of modern technology from the advanced countries also calls for much adaptation to suit the needs and local conditions of underdeveloped regions. To profit by the experience of the advanced countries and yet to introduce modern technology and modern social and political institutions in a way suitable to the particular stage of development of the country is a matter of crucial importance in the process of modernisation. Ultimately, success would depend on the growth of a rational outlook and the experimental attitude of mind, first, among the leadership at all levels and then gradually among the general mass of the people.

8.2. It is extremely important that advanced countries should help and encourage in every way all progressive groups within the country in promoting the process of modernization and refrain from offering technical or economic aid in any way which would hamper the social and scientific transformation.

THE SCIENTIFIC BASE OF ECONOMIC DEVELOPMENT

1. Phases of Economic Development

1.1. The essential characteristic of an underdeveloped country is an extremely low level of living, that is, inadequate supply of food, clothes, housing, drugs and other consumer goods, and also lack of facilities for education, care of health, social security, cultural amenities, etc., for the nation as a whole. It is possible to make available small quantities of consumer goods by direct imports or by domestic production, on a small scale, with the help of imported machinery. In most of the underdeveloped countries it is, however, not possible, for lack of necessary foreign exchange, to import or to produce, with imported machinery, enough consumer goods for the people as a whole. In India, the first textile mill was established in 1817; and India gradually became the second biggest producer of textiles, next only to America. One hundred and fifty years later, India would still remain underdeveloped. The production of textiles or small quantities of other consumer goods for a small part of the nation cannot, by itself, lead to industrialisation and economic development.

1.2. Economic development can occur only by increasing the per capita production of the nation as a whole, through an increasing use of machinery driven by steam or electricity as a substitute for human and animal labour. In countries with appreciable natural resources, it is necessary to establish the basic engineering and power industries to enable the manufacture of both consumer and capital goods within the country.

Establishing a minimum complex of such basic industries would take at least ten o fifteen years, for which planning must start ten or fifteen years in advance.

1.3. To increase modern industrial production would call for an increasing supply of engineers, technologists, and technical personnel. The only way to ensure this would be to establish and increase the number of schools, training colleges and universities, and also to train teachers for such institutions. This would take at least fifteen or twenty years; so that planning for this purpose must start fifteen or twenty years in advance.

1.4. The best way of utilising the raw materials and natural resources available within the country, for both domestic consumption and for exports, can be found out only through applied scientific research (1). Applied research, in its turn, must be based on advances in fundamental research. Also, to establish an adequate base for applied research it is necessary to promote the spirit of pure research and supply the stimulus of scientific criticism. This would be possible only when at least a certain minimum number of scientists are engaged in fundamental research, and opportunities for pure research are becoming increasingly available. It is therefore necessary to promote the advancement of both applied and fundamental research. To establish a minimum base for scientific research would take more than a generation of twenty-five or thirty years; this, being the most slowly maturing sector, must be given the highest priority.

(1) Even the most advanced countries are obliged to devote large resources to research for the improvement of products already being manufactured and also to develop new products in order to hold their position in the world export market. It is not possible for the underdeveloped countries to start or expand the export of fully or partly manufactured products by simply borrowing the current technology from advanced countries; it is essential also to develop applied research for a continuing improvement of technological methods.

2. The Scientific Base of the Advanced Countries

2.1. The scientific base of the modern age can be appreciated by even a brief review of the recent history of the advanced countries. Four hundred years ago the generally accepted view was that the earth was at the centre of the universe; the position of human beings was unique and supreme; and the highest sanction of truth was either divine revelation or abstract logical reasoning in the mind of man. In the sixteenth and the seventeenth centuries, there was a complete revolution in the picture of the physical world; the earth was seen as a small planet moving round the sun; and the method of empirical observations and experimentation was gradually established in both physical and life sciences.

2.2. Progress was at first slow in the sixteenth century. A few selected names may be recalled to indicate the gradual transformation of ideas. In astronomy, Nicholas Copernicus (1473-1543) supported the view that the planets including the earth itself were revolving in orbits round the sun; Tycho Brahe (1546-1601) supplied astronomical observations of unprecedented accuracy to make the next steps possible; Johann Kepler (1571-1630) formulated the descriptive laws of planetary motion; and Galileo Galilei (1564-1642) made conscious propaganda in favour of the new philosophy of the universe. In anatomy, Andreas Vesalius (1514-1564) published his observations on the human body in 1543; in physics, William Gilbert (1544-1603) gave an account of magnetism based on trustworthy experiments in 1600; in physiology, William Harvey (1578-1657) described the circulation of the blood in 1628; John Napier (1550-1617) supplied a convenient tool for computation by the use of logarithms; and Rene Descartes (1596-1650), a philosopher, contributed the powerful concepts of coordinates for geometrical representation and of mathematical functions. Francis Bacon (1561-1626), firmly stated that the only true method in science was to proceed from particular sense observations to wider generalizations (*Novum Organum*, Book I, xix), and clearly recognised that "the true and lawful goal of the sciences is . . . that human life be endowed with new discoveries and power."

2.3. The concept of an objective world of physical reality gradually took firm shape in the seventeenth century in the hands of gifted astronomers, mathematicians and scientists. A few names may be mentioned from among those who were born in the first half of the century: PIERRE FERMAT (1601-1665), CHRISTIAN HUYGENS (1629-1695), BLAISE PASCAL (1623-1662), ROBERT BOYLE (1627-1691), JOHN RAY (1627-1705), ROBERT HOOKE (1635-1703), ISAAC NEWTON (1642-1727), and GOTTFRIED WILHELM LEIBNIZ (1646-1716). The rate of advancement of science increased progressively in the eighteenth and the nineteenth centuries, and during the last few decades has opened new frontiers with almost unimaginable possibilities.

2.4. The advancement of science prepared the ground for the industrial revolution in Europe in the eighteenth century, first in spinning and weaving, next in the use of iron and steel, and then of electricity in the nineteenth century, which stimulated the growth of the capitalist economies in West Europe and North America. The spread of the scientific outlook also prepared the ground for the age of reason and the French revolution, which occurred at the end of the eighteenth century, and promoted the growth of nationalism in Europe, in its modern sense, in the nineteenth century.

2.5. The industrial revolution increasingly replaced human and animal power by steam or electricity to drive machinery for the increasing production of both consumer and capital goods. The development of engineering techniques led to a close linkage between science and technology; and during the last hundred and fifty years, industrial development is being stimulated by a scientific discovery or a scientific discovery is being stimulated by industrial needs.

2.6. For the last five or six thousand years, or more, the average per capita production remained more or less constant or fluctuated within narrow limits. The industrial revolution changed all this, and led to a spectacular increase in the variety and volume of goods produced. As a consequence of such increasing production, the standard of living of the advanced countries of West Europe and North America reached a level far higher than the rest of the world. Also, the advancement of science, technology and industry, made it possible for the western countries to become strong military powers; and, because of such military supremacy, the west was able to bring a large part of the world either into direct colonial rule or into conditions of economic or political subjugation.

2.7. The last forty years have also seen the rise of U.S.S.R., as another world power, rapidly growing, through the promotion of science and technology, in economic, industrial and military strength together with a continuing increase in the level of living. The monopoly of scientific and technological knowledge and the unchallengeable military supremacy of the western countries have now gone. The increasing parity between the "western" and the "eastern" countries in science, technology, industry, and military power is a most significant fact of the present time. Because of the unprecedented destructive power of atomic and nuclear weapons, it has become absolutely necessary to avoid a nuclear war which would be catastrophic for both sides and the whole world. Coexistence of both the "western" and the "eastern" powers has become indispensable.

2.8. There is no intention on either side to make a direct attack. The advanced countries pose no special problems because it is not possible to hold such countries indefinitely in subjugation. However, so long as there are underdeveloped areas, both power groups are likely to try to extend their influence over the less advanced countries; and this would remain a continuing source of potential conflicts. The very existence of underdeveloped countries should, therefore, be seen as a threat to peace. Rapid transformation of all the underdeveloped countries into modern viable societies is an essential condition for peaceful coexistence. Such a transformation would promote the enlightened self-interest of both power groups, and would also create conditions favourable for the advancement of human and cultural values on a world-wide basis.

3. The Role of Science in the Modernisation of the Less Advanced Countries

3.1. Modernisation of the less advanced countries through rapid industrialisation is thus an urgent need of the whole world. Is such modernisation possible or can a modern society with a viable economy, with expanding social and political freedom, and cultural amenities, be sustained without establishing a sound scientific base? This is a question of crucial importance for the present age.

3.2. In order to answer this question, it is necessary to appreciate the deeper changes in human thinking which were brought about by the emergence of science. In every sphere or organised activity in human society, authority has always been associated, and must always be associated with a system of hierarchical levels. This applies to primitive societies, matriarchal, patriarchal, or tribal; successive levels of feudal lords; organised churches and religions; military, police and administrative systems; enterprises, business and commerce; and law. A law court of appeal may reverse the decision of a lower court; but the decision of the court of appeal is itself subject to change by a still higher court. The decision of the highest court, to which a case has been actually referred, has to be accepted not because such a decision is necessarily right, but because it is the decision of a superior authority (2). Society must accept this authority principle for stability and orderly progress, even in organised revolutionary activities.

3.3. This very authority principle must, however, be absolutely and completely rejected in the field of science. Modern science is based on a patient accumulation of facts, on the study of processes and their interrelations or interactions and a stability or uniformity of nature (3) which can be discovered by the human mind. The findings of the most eminent scientists are subject to critical check by their professional colleagues and by the youngest scientific workers, and must be rejected if there is no satisfactory corroboration. Science can advance only through free criticism on a completely democratic basis, with every research worker of competence enjoying equal status. The theoretical or conceptual framework of science must be

(2) It is possible, indeed, that this decision itself would have been reversed if there had been a still higher court to which the case could be referred. If a decision of a higher court of appeal is considered to be like the turning up « heads » (in tossing an unbiased coin) when the decision upholds the verdict of the lower court, and is considered to be like the turning up of « tails » when the verdict of the lower court is reversed, then the successive decision of the higher court would look like the results of the tossing of a coin. This would be the real guarantee that the system of law is functioning properly.

(3) The phrase « uniformity of nature » must be, of course, interpreted to include chance events and random processes. Although games of chance were known and were widely prevalent in ancient times in China, India and other countries, it is important to note that the concept of probability did not arise until the 16th and the 17th centuries, that is, not until the emergence of modern science. This in easy to understand. Before the emergence of the modern scientific view of an objective world of physical reality, all chance events would have to be necessarily ascribed to the whims of gods, demons, or supernatural forces. After the emergence of the scientific view of an objective world of physical reality, it became necessary, both logically and psychologically, for the human mind to accommodate the occurrence of chance event as an integral part of the uniformity of nature. This could be accomplished only on the basis of the theory of probability, or rather, as I should prefer to put it, only through a statistical view of the world. It seems to me, therefore, that the concept of probability, of the statistical view of the world did arise at the same time as the emergence of modern science only because it could not possibly have arisen earlier.

continually revised to find a proper place for all known facts. A single new observation may call for a more comprehensive theory. The older accumulated knowledge continues to remain valid; later discoveries must, however, be integrated with the earlier knowledge. The accumulation of scientific knowledge is inscreasing through the efforts of all the scientific workers of the world. A new fact may be observed or a new theory formulated by any worker, however young, and in any country where research has been established. International collaboration is, therefore, an indispensable condition for the progress of science.

3.4. Authority derived from status is irrelevant to science. Science has introduced a new concept of "scientific", or "objective validity" which has its foundation in nature itself, and which cannot be upset by any authority based on status or by supernatural powers. The transformation of all the advanced or rapidly advancing countries has been based on accepting, in an increasing measure, a scientific or rational view of life. This is the foundation of the modern age.

3.5. It is essential in every country to establish and strengthen the outlook of science, a way of thinking which becomes more and more powerful as it is more widely adopted, and which replaces dogma, superstition, and outdated customs. This scientific outlook cannot be established by force. It must depend on acceptance through proper understanding. In practical affairs, the important point is that a wise policy and programme of action should be increasingly adopted on the basis of rational argument, supported by relevant factual evidence, and should not be rejected because of emotional bias or formal dogmas or conventional rules of procedures. It is, therefore, necessary continually to encourage and promote the advancement of science in every country, large or small. Because science is indivisible, and also because science must be established in every country, it is also necessary, continually, to promote scientific collaboration between all countries of the world, large and small, and advanced or developing.

3.6. It is scarcely necessary to point out that there is no conflict between the scientific and rational view of life, on one hand, and aims and objectives based on moral or cultural values, on the other hand. On the contrary, moral and cultural values which are truly universal, and are or narrowly sectarian or nationalistic in a restricted sense, must have an objective and rational basis.

3.7. The advancement of science and the growth of the scientific outlook must be recognised as an essential condition for the modernisation of the less advanced countries. It is necessary for each country to have, as quickly as possible, a sufficient number of men with a scientific outlook to influence the thinking of the nation. How to attract and hold a sufficient number of able persons to science is thus the crucial problem of national and world development. This can be achieved only through a proper and adequate social appreciation of science and scientists. The actual transformation must be brought about from within each country. Scientific aid from the advanced countries can, however, be of great help in this process.

4. Present Programmes of Technical Aid

4.1. The need of technical aid has been recognized for some considerable time. Bi-lateral or multi-lateral and international technical aid has often taken the form of either offering educational and training facilities to young workers from the less advanced countries or sending technical or scientific experts to such countries. Considerable benefit has no doubt accrued through such aid but it is necessary to recognise that much effort has also been wasted.

4.2. Scholars from the less advanced countries are usually selected on the basis of results of examinations; success in examinations not being a necessarily reliable indicator of scientific or technical ability, the very process of selection is inefficient. Some of the young scholars have difficulty in adjusting themselves to the pattern of living in the advanced countries. Some of them do not do well in their studies. Some pass the examinations successfully but have no aptitude for scientific work. Some of the more able scholars prefer to live and settle down in the advanced countries, especially in the U.S.A., because of the higher level of living or greater opportunities for scientific work. Some scholars of ability, when they return to their own countries, are unable to find suitable openings for a scientific career; and some of them go back to the country where they were trained. In applied science and technology, and especially in social sciences, many young scholars, who had often studied problems or learnt methods which are appropriate for advanced countries but totally irrelevant to their own native countries, are unable to adapt or develop methods to suit local conditions. Out of the large number of scholars who go to advanced countries for training, only a very small number of really able scientific workers ultimately become available for fruitful work in their own country. The cost of giving scientific or technological training in an advanced country is also very high. Giving training to individual scholars in advanced countries (whether the expenses are provided in the form of foreign aid or met by the scholars themselves or by the country of origin) have been, therefore, extremely wasteful in terms of both men and money.

4.3. There has been also continuing difficulties in finding suitable individual experts for the less advanced countries. Competent scientific workers are reluctant to accept such assignments partly because of the lack of facilities for their own work in the less advanced countries and partly because their scientific or academic career is likely to be adversely affected through their absence abroad. In consequence, assignments sometimes have to be given to persons who are not fully qualified for the job, with unsatisfactory results. To create suitable conditions for scientific work in the less advanced countries is an indispensable condition for attracting competent scientists to go out to such countries.

4.4. Programmed technical aid on a group basis has been more effective. A team of young engineers from a less advanced country can receive most valuable training in an advanced country when such training is oriented to specific technological projects. Teams of experts from advanced countries have also been of very great help in establishing factories or in starting new projects in the less advanced countries. Such technical aid, especially in engineering, technology and applied sciences, should be continued and expanded. Special projects for establishing technological and research centres in the less advanced countries have also been taken up by some of the international agencies. This type of aid can be of great value provided a sufficient number of scientific workers in the less advanced countries can be trained to work in such centres, and also provided necessary conditions are established to enable them to do their work properly.

5. Scientific Education and Research

5.1. It has been argued in the earlier sections that for modernisation it is necessary to establish a foundation for scientific research and the social appreciation of science in the developing and less advanced countries. Every path-finder in a new field of research must work in the first instance by himself; if he is successful, other persons gradually get interested in the subject. Such path-finders always had, and will always have to overcome much opposition, and even hostility, until the new subject becomes a recognised part of the « established » field of science. But it is only a few scientists of outstanding ability who can work in isolation. Most research workers require the stimulus of free interchange of views and ideas and of appreciation among professional colleagues.

5.2. The community of scientists has a structure of a series of widening circles similar to the structure of scientific subjects or of science as a whole. When a top scientists speaks appreciately of some work in his special field, other scientists or lay men accept his evaluation and pass on the information to others. The social appreciation of science gradually emerges as a result of the diffusion, in widening circles, of the views of scientists, who are experts in specialised fields of research, to scientists in related and associated fields, then to scientific workers generally, and finally, through persons of position and standing who have contacts with scientists, to the general public. The speed with which such appreciation can spread increases rapidly with the increase in the number of scientific workers and improvements in the channels of communication. In the advanced countries, the awareness of the importance of science is increasing rapidly which, in its turn, is raising the social status of scientists and is promoting an increasing flow of resources for research.

5.3. The whole process is extremely slow in underdeveloped countries. The number of research scientists is very small, and channels of scientific communication are non-existent or meagre. Scientific workers usually receive lower pay and have a lower status than the administrative staff in government or in business concerns; and have to work in a rigid system of hierarchical authorities. Promotion may depend, not so much on the high quality of the scientific work done, but on success in pleasing those who are higher up in the official hierarchy. Even permission to apply for posts elsewhere is subject to the discretion of superior officers. There is a continuing tendency to bring scientists and scientific work under stricter control of the administrators, partly, perhaps. from an unconscious fear of rivalry of power. Even if the right of criticism is accepted in principle, it is restricted in practice because scientific workers are often afraid, rightly or wrongly, of giving offence to persons holding higher posts. In consequence, many scientists in underdeveloped countries suffer from a lack of self-confidence, and are afraid to take up original lines of investigation. There is little possibility of a proper evaluation or appreciation of scientific work within the country. This leads to an exaggerated dependence on the opinion of foreign scientists and gives rise to much imitative work. Also, when there is lack of appreciation or criticism from the advanced countries, there is sometimes a tendency to ascribe the unfavourable view to racial or national prejudices, and there is resistance against collaboration with foreign scientists.

5.4. In underdeveloped countries there are very few, sometimes only one or two, individuals of outstanding ability in scientific research or in any other scientific field. As leadership can be supplied only by individuals of high ability, and as such persons are few in number, it is much more difficult in underdeveloped countries to utilise the services of individuals of average ability and qualification. The advanced and advancing countries have a double advantage. They have a large number of persons with qualities of leadership and can, therefore, utilise in a fruitful way larger numbers of persons of average ability. This is why many scientific workers from underdeveloped countries, who are unable to do much useful work in their own native country, can often do very good work in the environment of a higher state of organisation of research in an advanced country.

5.5. The aim of scientific aid must be to create in every underdeveloped country, as quickly as possible, a sufficient number of research scientists to form a community of professional workers which would be sufficiently large to facilitate an independent evaluation of scientific work through free criticism and frank exchange of views. It is, therefore, necessary to focus attention on identifying and giving support to persons who have the ability to undertake research work of high quality, and to try to increase their number as quickly as possible, and at the same time to offer opportunities for training to persons of average ability whose services would be equally essential in supplying a wide base for the pyramid of scientific work.

5.6. There is urgent need of fostering the spirit of objective scientific criticism through free expression and exchange of views and opinions. One effective way of promoting this would be to make it easy for scientific workers to migrate from one post to another and give an absolute guarantee of such freedom to migrate. Any scientific worker who feels, rightly or wrongly, that he has not enough opportunities for fruitful work in one institution would be free to migrate to some other institution. Such migrations or the possibility of such migra tions would have an indirect but most important selective effect on scientist at all levels.

5.7. It is necessary to recognise that the social value of an individual scientist of high ability is far greater in a developing country because of the leadership he may be able to supply. It is only scientists engaged in fundamental research who can function as the eyes and ears of the nation in making the nation appreciate and identify urgent needs of applied research. The emergence of even one or two outstanding research scientists can enhance the prestige of the nation in a most significant way at the international level and promote the growth of self respect and self confidence of the nation. This is why it is particularly important in developing countries to identify such individuals, at first very few in number, and give them all possible facilities and encouragement to continue their work in their own country.

5.8. In the highly developed countries science advanced both from progress at the highest levels of research, at the top, and from the wide diffusion of education, at the bottom. The same strategy may be adopted with advantage in the less advanced countries. What is urgently needed is to lay the foundations, with as wide a base as possible, for a country-wide system of school education oriented to science and, at the same time, to develop advanced studies of science and technology and research at the highest level. The school system must fit into the economic life of the general mass of the people and have its grass roots in the villages. It must offer facilities for training technicians and technical personnel for science and technology and also supply candidates of outstanding merit for admission to higher scientific and technological institutions.

6. Need of Direct Aid for Science

6.1. I shall offer, briefly, a few suggestions for giving direct aid for the development of science in the less advanced countries. I have stressed the need of building up a system of school education with a definite orientation to science. It would be, however, a fatal mistake to establish an expensive system of education on the model of the advanced countries which would have little relevance to local needs and would be beyond the means of the national economy. It is necessary to evolve a system, through experimentation and trial and success, which would be within the means of the national economy. The approach must be therefore to use teaching aids which are easily available or can be made available on a large scale and at a low cost. As most of the pupils will be living in villages, it would be of great advantage if agriculture and some of the rural industries can be adopted as a base for the teaching of science. The programme may consist largely of nature studies, observations, and experiments which can be done with the help of simple articles, specimens, etc., likely to be locally available or which can be constructed with local materials.

6.2. There would be still some need of supplying teaching aids and materials from outside which would have to be specially designed to reduce costs. It is essential also to prepare books of instructions and text books to suit a fairly wide range of needs. These are difficult tasks which would call for extended study and research by scientists of high calibre with a serious interest in problems of science education. As basic conditions in underdeveloped countries are likely to be similar to a large extent, it may be possible to evolve broad general methods for science education which would be capable of being adapted without much difficulty to suit differing local conditions.

6.3. A great deal of pioneering research would be necessary for this purpose for which the help of advanced countries is indispensable. A good deal of experimental studies will have to be undertaken under conditions actually prevailing in underdeveloped regions. In the beginning, the studies would have to be organised on a small scale with the help and support of the local authorities and of such teachers and scientists as may be available to cooperate of the local authorities and of such teachers and scientists as may be available to cooperate in the venture in the underdeveloped country itself. The project can be gradually extended, in the light of experience, to cover different subject fields at different educational levels, and also from one underdeveloped country to another. Fortunately, even one or two scientists can start the work in one single country. The important point is to make a beginning at the earliest opportunity.

6.4. I may now mention a second type of programme. Certain facilities for scientific research are already available in India and other developing countries. In most of these countries, scientific work is being hampered for lack of small replacement parts, additional accessories and instruments, and supply of essential consumable stores which have to be imported from the advanced countries. It is often difficult to secure import licences on account of shortage of foreign currency. This difficulty can be overcome through a simple plan of gifts in kind of replacement parts, instruments and equipment, stores, books and journals and reprints or microfilms of scientific papers etc., to be arranged through non-governmental committees of scientists. Such committees, which can be set up in the advanced countries through or in cooperation with appropriate scientific organizations or societies, would try to secure suitable grants from Government and other sources. In developing countries where scientific research has already started, the counterpart committees of scientists would also be set up, preferably, at a non-governmental level and with a majority of members from universities and non-governmental scientific institutions. All arrangements would be made with the concurrence of the government of the less advanced country concerned, but decisions relating to gifts for scientific work must be made by direct consultations between the scientific committees themselves. A scheme of this type can be usefully started, on an experimental basis, for a few selected countries, at a low cost, with gifts to the total value of perhaps one or two hundred thousand dollars per year. The amount can be increased if the experiment proves successful.

6.5. Another important form of scientific aid would be to arrange for competent research scientists from the advanced countries to work for a year or two in existing research units in the less advanced countries or to help in establishing high level research units in such countries. The less advanced countries can offer challenging problems and opportunities for research in many fields of science, which cannot be duplicated in the advanced countries, for example, in geology meteorology and geography; biology, botany, and zoology; agriculture; medical science and public health; economics of development; linguistics, archaeology; and historical and cultural studies of various kinds. In some of the developing countries there would be also increasing opportunities for active participation in research in mathematics and statistics, and physico-chemical and technological sciences. In establishing research units in underdeveloped countries it would be desirable to keep one broad aim in view, namely, to encourage joint studies by active collaboration between different research units. This would help in developing a community of research cells or units which, in its turn, would foster the growth of the spirit of scientific criticism and appraisal among wider circles of scientific workers.

6.6. To attract competent visiting scientists it is necessary to offer them facilities to pursue or start fruitful research in the less advanced countries; sometimes special equipment may have to be provided for this purpose. Secondly, the assignment in a less advanced country, would have to be treated as deputation in the same way as participation in scientific expeditions, and which would be recognised as a part of normal duties and also as a possible qualification for promotion. The visiting scientist must receive sufficient compensation in his home currency to meet his continuing home commitments during his absence abroad. Living and other local expenses should be normally met by the institution or by the government of the country in which he would work. Such sharing of costs would promote effective cooperation in the less advanced country, and would also reduce the total cost appreciably.

6.7. An important part of the responsibilities of a visiting scientist would be to give training to the scientific workers of the underdeveloped countries. When necessary, the visiting scientists would be able to select, for further training in an advanced country, the right type of persons who can be depended upon to go back to their own country after the completion of the training abroad. It would be also possible to give aid in the form of equipment and instruments in an effective way on the basis of objective appraisals of needs and possibilities by the visiting scientists.

6.8. A fourth programme could be to send from advanced countries young scholars, who have just finished their education in universities or higher educational institutions or have already done some research, to start or continue suitable lines of research for about two years or so in existing institutions or in research units to be established for this purpose in underdeveloped countries. The common participation in research projects of young scholars from the advanced and the underdeveloped countries would be of great help in establishing scientific traditions and an atmosphere of scientific criticism. It would promote self-confidence among the scientific workers of the underdeveloped country, especially, if the visiting scholars from advanced countries take higher degrees from institutions in the less advanced countries.

6.9. All the above forms of scientific aid can be started, on a small scale and at low cost, and, if successful, can be expanded in the light of experience. Also, these forms of scientific aid would not in any way overlap or hamper bigger programmes for gifts of expensive equipment or large projects for the setting up of national or regional centres and institutes for scientific research in the less advanced countries. On the contrary, the modest programme described in this note would prepare the ground for bigger projects.

7. Conclusion

7.1. In conclusion I may refer, very briefly, to some recent developments. After the second world war the movement for terminating colonial rule gained rapidly in strength, and one country after another in Asia and Africa has won political independence. It is being increasingly realised, however, that independence is not enough for economic development. The need of economic and technical aid is also being increasingly appreciated. Both the « western » and the « eastern » powers have started helping in the economic development of the less advanced countries in Asia, Africa and Latin America, but still without an adequate impact. The time has come to recognise that economic aid is essential but is also not sufficient.

7.2. Revolutions to capture political power have been occurring throughout human history and are even now occurring in many of the politically independent countries in Latin America or in most of the newly independent countries in Asia and Africa. Such revolutions do not automatically promote rapid economic development, because purely political revolutions do not lead to any fundamental transformation of the old society based on the principle of authority associated with levels of status. It is becoming increasingly clear that rapid economic development cannot be achieved without developing a structure of society in which decisions would tend to be made more and more on grounds of reason, that is, in accordance with the principle of objective validity instead of authority. It is relevant to note that the French Revolution was preceded by the age of reason; the American War of Independence had the support of influential leaders inspired by the spirit of science; and the socialist government, which was established after the October Revolution in 1917 in Russia, made great efforts to build up a countrywide system of science-oriented education and to promove scientific research and, in this way, succeeded in modernising the whole society leading to rapid economic development.

7.3. One thing is clear. In the absence of rapid economic development, political conditions in the less advanced countries would remain unstable. In many or most countries there would be one revolution after another tending to get the two power groups involved directly or indirectly in the struggle. The world must get out of this vicious circle. There are only two possibilities. One is for a violent type of revolution to occur which would suddenly change the whole structure of society to make it fit for rapid development of science and economic progress. The other alternative is deliberately to build up the foundation of science-oriented education and research to promote the modernisation of society in a peaceful way, and make conditions favourable for economic development.

7.4. Aid for scientific and economic development from either the western or the eastern countries, even when given in a spirit of competition, would be cooperative in effect. In any event, competition in constructive tasks of building up scientific foundations in developing countries is less dangerous and is likely to be far more useful than competition in the methodologies of warfare. Also, collaboration in promoting education and research in pure science can be pursued without any threat to national security or national interests, and would be of great help in promoting a rapid advance of the underdeveloped countries and in fostering better understanding among the nations of the world. The advanced countries have a great opportunity for peaceful cooperation in giving aid for science.

This pamphlet has not yet been published in any journal and is being printed for private circulation. Certain aspects of these problems were discussed by me in articles and addresses between 1955 and 1959 which were reprinted in *Talks on Planning* (1961), and in other articles such as *A Note on Problems of Scientific Personnel* (1959), *Recent Developments in the Organisation of Science in India* (1959), and a lecture at Sofia University in December 1961. Some of the ideas given in the present pamphlet were presented by me before a Conference on International Cooperation in Salzburg-Vienna in July 1962.

Professor P. M. S. Blackett in his presidential address to the British Association for the Advancement of Science in 1957 and in other articles in *Nature*, (3 February, 1962; May 1962 etc.) has considered various problems from the point of view of the advanced countries. Professor Stevan Dedijer made a penetrating analysis in article in *Nature* (6 August, 1960) and in another article published recently in Stockholm, *TVF*, 33, (1962)

STATISTICAL TOOLS AND TECHNIQUES IN PERSPECTIVE PLANNING IN INDIA

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Introduction

The phrase 'perspective planning' is being used in India since about 1954 or 1955 in the field of National Planning in which long range targets have to be set up 10 or 15 or 20 years in advance. The object of the present paper is to explain why perspective planning is essential in the case of under-developed countries and give some illustrative examples of the statistical information and methods which have been found useful for this purpose in India. This is not the occasion to attempt a comprehensive discussion of techniques of perspective planning.

It is useful to make a distinction between projections and targets. The word 'projection' is used in the same way as in advanced countries to refer to the value of production, or of consumption or of other variates, at a specified date in future, estimated on the basis of historical records. Projections are essentially estimates obtained on the basis of analysis of time series or some kind of extrapolation in time. It is convenient to use the word 'target' as the value of production, of consumption, or of other variates of interest which is desired to be attained on a specified date in future, through the process of implementation of an economic plan. The word 'target' would be used consistently in this sense.

Objects of planning in India: The ultimate objects of planning are to improve the level of living, and expand facilities for education, care of health, cultural amenities etc. for *all* the people of the country. A spectacular improvement in the level of living of the advanced countries has been possible in the past, and a similar improvement would be possible in the less advanced areas in future, only through a continuing increase in the per capita production of *all* the people of the country. Such increase in per capita production can be attained only through a continuing substitution of human and animal power by machines, driven by steam or by electricity, for productive purpose of all kinds including industry, agriculture, transport and distribution.

Changes in the level of living: As our chief concern is with the improvement of the level of living, a continuing National Sample Survey was started in India in 1950 which is collecting comprehensive information on various aspects of the level of living in rural and urban areas with a view to assessing the change over time. The total per capita expenditure per month on all consumer goods and services of each household has been used as a rough indicator of the level of living of the household. The method of fractile graphical analysis (1) has been used to study the distribution by size of total per capita expenditure per month of households. Studies are also being made of the relationship between the total per capita consumer expenditure and the per capita consumption of individual items in terms of money and also in physical quantities where possible.

A study of the distribution of per capita total consumption expenditure by decile groups of households (arranged in ascend-

(1) See *A method of fractile graphical analysis*, « Econometrica », vol. 28, pp. 325-351, 1960; also *A preliminary note on the consumption of cereals in India*, « Bulletin of the International Statistical Institute », vol. 38, pp. 53-76, 1962.

ing order of the total per capita consumer expenditure in each household), shows that for the data collected in the National Sample Survey Round 8 (covering the period July 1954-March 1955), the percentage share of the lowest decile group was 3.01 in rural areas and 2.65 in urban areas, and of the second lowest decile group (between the tenth and the twentieth percentiles of households ranked by per capita expenditure) was 4.09 and 3.90 for rural and urban areas respectively. For purposes of perspective planning, four per cent may be used as the share of the second lowest decile group of households (2).

Targets of planning: The average per capita expenditure in the second lowest decile group was a little over Rs. 10 per month (3) in 1960-61. For purposes of illustration, it is possible to adopt a target of raising, over a period of 15 years, the average per capita consumption expenditure in the second lowest decile group of households from Rs. 10 to Rs. 20 per month (or fifty dollars per capita per year). This amount, at 1960-61 prices, would provide only a very modest level of living in terms of food, clothing and other essential goods or services and amenities.

Doubling the per capita expenditure in fifteen years implies a rate of growth of nearly five per cent per capita per year. It is of interest to note in the present connexion that the per capita income in USA has increased sevenfold in the cours of 120 years or at a rate a little over 1.6% per capita per year. A reasonable target of planning in India would thus call for a rate of increase of income at a rate nearly three times greater than the actual rate of increase attained in the USA during the last 120 years. The above comparison would supply a rough idea of the dimension of the efforts required for economic development in India.

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(2) The lowest decile group has not been used because it may be a somewhat heterogeneous category comprising vagrants, persons living in isolation, tribal people, households in a transient income group etc. many of whom would require special ameliorative measures.

(3) One rupee = 1 shilling 6 pence = 0.21 U. S. cent approximately.

For purposes of planning it is necessary to deal with actual figures and not merely in percentages. The population of India is expected to increase to about 650 million compared with about 430 million in 1960. (This, of course, is the population projection for 1975 on plausible assumptions, and not a target; in fact, if it were possible to bring about a reduction in the rate of growth of population, Indian planners would no doubt adopt a much lower figure as a target). The number of second lowest decile group of households in 1975 would be about 12.5 million on the basis of about five persons per household. To attain a target of Rs. 1,200 (or $ 240) per year per household, the aggregate income of the second lowest decile group of households would have to be Rs. 15,000 million. If it is assumed that this group would still continue to have a four per cent share of the total expenditure of the households (4) then the aggregate national consumption expenditure of households would be 25 times greater or Rs. 375,000 million. The aggregate national income of India in 1975 would have to be somewhat larger to allow for investments and certain other items. The level of national income to be attained in 1975 would have to be somewhat more than double the target of income at the end of Third Five Year Plan in 1966. The rate of growth would have to be about seven per cent per year.

Need of rapid industrialization: Such a rapid change (at a rate three times greater than that of USA) would call for rapid industrial expansion over a period of 15 years.

The ultimate aim is expanding continually the production of consumer goods and services. It is necessary to increase the supply of machinery and energy for this purpose. In

(4) In India the distribution of consumption expenditure of households by size of expenditure has been found to be steady (with some small fluctuations probably due to the effect of changes in prices) over the last ten years. The pattern of distribution of income of households by size of income has also been found generally to change only very slowly over time in most countries of the world. The assumption that the share of the second decile group (or of other fractile groups) of households would remain practically the same in India in 1975 is plausible.

India, and in most of the other underdeveloped countries, it is not possible continually to import machinery for production of goods or of fuel on account of shortage of foreign currency. It is essential to establish and expand industries to manufacture machinery, electricals, transport and construction equipment. To increase the capacity for the production of capital goods and energy would be thus the only sound foundation for the expansion of consumer goods and services in future.

At the same time, in all underdeveloped countries it is possible to increase the production of consumer goods with small tools by using traditional methods. This type of production is labour intensive and would give gainful employment to a large number of people who would otherwise remain idle for a good part of their time.

In India a dual strategy was adopted from 1956 in the Second Plan to expand, on one side, the strategic heavy industries for steel, metals, machinery, electricals and chemicals, etc. to build up the foundations of industrial progress, and at the same time also to expand the traditional cottage industries and small scale production.

Targets of capital goods: It is therefore necessary to expand and set up not only targets of income or of consumer goods but also of machinery, steel and other metals, electricity, transport, etc. which would be used for the production of the desired volume of consumer goods and services.

Targets of scientific and technical personnel: To achieve the targets of production, it would be necessary rapidly to increase the technical staff to prepare and implement an increasing number of projects. Training facilities must be expanded sufficiently quickly to turn out technical and scientific personnel in adequate numbers at all levels. Scientific and technological research would have to be expanded and oriented to serve the needs of national development in an effective manner. Fundamental research as well as training in research must also be encouraged and developed at the same time to foster the accumulation of basic knowledge and to supply a sound foundation for national decisions being made increasingly on rational grounds.

Balances at the stage of production and utilization: An essential condition for successful planning is to estimate in real terms the requirements of each project to ensure that right quantities of materials, machinery and men are available at the right time at every stage of the implementation of the project. Also, products and services resulting from the completion of each project must be promptly and effectively utilised to promote the execution of other projects and for the progress of the plan as a whole.

The physical targets of production must be balanced in terms of physical quantities of raw materials, machinery, energy, transport etc., and also in terms of man power and of the flow of money. Incomes are generated in the very process of production; and supplies are utilised through market operations. Planning requires that aggregate incomes should be balanced with expenditure, savings should match investments, and the supply and demand of individual goods and services should be balanced in real terms so as to avoid any inflationary rise of prices or undesirable shifts in prices. Physical and financial planning are different aspects of the same reality.

In India a perspective view of development over a long period of years began to be taken from the end of 1954. It was recognised that the targets and the balances of materials and of man power would be only approximate partly for lack of information and partly for defects in organisation and implementation. It was therefore recognised that planning would have to remain flexible and to enable necessary adjustments being made almost continuously. At the same time it was essential to keep in view a wide time horizon of 15 or 20 years or more.

The use of simple models: In 1954-55, some simple models were used to work out the basic strategy of the Second Five Year Plan. The total investment was divided into two parts, one λ_i as the fraction used for investments for the production of capital goods, and the other λ_c as the fraction used for investments for the production of consumer goods ($\lambda_i + \lambda_c = 1$). If the corresponding net output-investment ratios for the production of investment goods and for the production of consumer goods respectively are β_i and β_c then the total net output-investment ratio is $\beta = \lambda_i \beta_i + \lambda_c \beta_c$. By using the following two sector model, and using numerical values for the total investment, and estimated values of β_i and β_c, suitable values of λ_i and λ_c were selected so as to enable the economy to grow at the target rate of five per cent per year or so. In order to estimate the volume of employment, the capital investment required per worker, say θ, was also used.

The growth of national income (Y) in the two sector model is given by the following formula:

$$Y_t = Y_0 \left[1 + \alpha_0 \frac{\lambda_i \beta_i + \lambda_c \beta_c}{\lambda_i \beta_i} \{(1 + \lambda_i \beta_i)^t - 1\} \right]$$

in which Y_o is the national income in the base year, Y_t the national income in the t-th year, and α_o the rate of investment in the base year.

On this basis, a Draft Plan-frame for the Second Plan was prepared in March 1955 (5). Values of the different parameters as used in the Draft Plan-frame, the Second Plan (1956-61) as actually realised, and the Third Plan (1961-66) as estimated, are shown in the Table given below.

(5) The methods used have been described in *The approach of operational research to planning in India*, « Sankhyā », vol. 16, pp. 3-130, 1955.

TABLE I

Investment allocation, capital per worker and net output-investment ratio

plan	percentage allocation of investment for		capital per worker (Rs.)	net output-investment ratio		
	invest-ment goods	consumer goods		invest-ment goods	consumer goods	total ($\lambda_i\beta_i+\lambda_c\beta_c$)
(1)	(2)	(3)	(4)	(5)	(6)	(7)
	λ_i	λ_c	θ	β_i	β_c	β
Second Plan:						
Draf plan-frame (1955)	33	67	5,100	0.20	0.67	0.51
Second plan: actual (1956-61) . . .	36	64	5,400	0.11	0.53	0.38
Third plan: estimate (1961-66) . . .	39	61	6,900	0.21	0.63	0.47

Many changes were made in the targets and allocations of the Draft Plan-frame at the stage of the preparation of the Second Plan; the values of the parameters of the Second Plan as actually realized and the values given in the Draft Plan-frame are therefore not strictly comparable. The interesting point to note is that the estimated parameters for the Third Plan are fairly close to the parameters used in the Draft Plan-frame.

The rate of investment (α_o) in the first year of the Second Five Year Plan was 9.8 per cent, the initial national income (Y_o) was Rs. 108.0 billion and the values of the other parameters were $\lambda_i=36\%$, $\lambda_c=64\%$, $\beta_i=0.11$, $\beta_c=0.53$, as given in the second row of the above table. Using these values in the above expression, the estimated national income comes out as Rs. 129.7 billion for 1960-61 against an actual figure of Rs. 130.1 billion both expressed at 1952-53 prices. In the case of the Third Five Year Plan, using the parameters given in the third row of the table, an initial income of Rs. 145.0 billion (at 1960-61 prices) and an initial rate of investment of 11 per cent, the estimated income for 1965-66 on the basis of the two-sector model is Rs. 188.9 billion against an estimate of Rs. 190.0 billion given in the Third Five Year Plan on the basis of detailed sector-wise calculations.

It may be concluded, therefore, that the two-sector model can supply a fairly reliable method for estimating future income. Values of the parameters used for the base period are no doubt subject to errors of estimation; but this would be true in the case of other methods also. The two sector model gives realistic estimates presumably because it has reasonably correct structural relations between relevant variables.

Values of output-investment ratios: Output-investment ratios β_i and β_c determine, together with the chosen values of λ_i and λ_c and the total amount of investment, the rate of increase of income and have an important role in planning. These two coefficients of net output-investment ratios were calculated from technological and statistical information in respect of hundreds of enterprises combined with appropriate weights. The calculated values for manufacturing industries are given in Table A 1.

Need of perspective planning: *Steel*: The need of looking a long way ahead was learnt in India through experience. I shall give one example. In 1949 when preparatory work had just started for the First Five Year Plan, a decision was practically reached to increase the capacity for the production of steel from a little less than oue million ton per year to two million tons per year in the course of five years. However, a careful survey was made of the current demand as in 1949. It was found that the maximum demand would be about 1.5 million ton per year. With marginal expansion of existing steel plants, it was possible to produce about a million ton per year within the country. Owing to the wide prevalence of the views of short-range economic theory, it was therefore decided that it would be inadvisable to include a new million ton steel plant in the First Five Year Plan of India.

In consequence, great difficulties began to be experienced from the early years of the First Plan. Practically all the estimates for investments had been made in purely financial terms and a sizable increase in investments had been approved purely on a financial basis. As soon as the investment projects began to be implemented, there was a sharp and continuing increase in the requirements of steel and other goods and services. Very soon the demand for cement increased to nearly three times the domestic supply. There was also a continuing and large expenditure of foreign currency for the import of steel, which added up to something like 2,000 million dollars in the next ten years or so. In 1950 it would have been possible to establish a new million ton steel plant with perhaps about 150 million dollars of imported machinery. Had this project been started at that time an additional supply of one million ton of steel (worth more than one hundred million dollars per year) would have been available from the early years of the Second Fide Year Plan, and would have resulted in a very large and continuing saving of foreign exchange. The decision to drop the million ton steel project from the First Plan was due to attention being focussed only on the current demand in 1949, that is, due to a complete failure to appreciate the need of looking ahead to get ready to meet the demand for steel which was certain to increase rapidly in future.

Targets of steel in 1970: At heavy cost we had learnt the lesson of not proceeding with the building up of capacity for steel production 12 or 15 years ago. Much attention is now being given to advance planning for steel. A detailed analysis of the requirements of steel is made, where possible, by individual items of production. With a given set of production targets for, say, 1970, it is possible in this way to prepare useful estimates of the requirements of steel. Some illustrative figures for the transport equipment industry is given in the following table.

TABLE 2

Steel requirements for transport equipment industry in 1970

industries	production target in 1970	tons of rolled steel required per unit of output	steel requirement in 1970 (in thousand tons)
(1)	(2)	(3)	(4)
1. Steam locomotives . . .	300	150	45.0
2. Electric locomotives . .	150	55	8.3
3. Diesel locomotives . . .	200	55	11.0
4. Wagons	40,000	12	480.0
5. Passenger coaches . . .	2,500	30	75.0
6. Automobiles	180,000	2.9	522.0
7. Motor cycles, scooters . .	150,000	0.1	15.0
8. Bicycles	4,000,000	0.02	80.0
9. Ships (GRT)	160,000	0.65	104.0

Source: Demand for steel, special steel and pig iron. India: 1960-1970: Perspective Planning Division, Planning Commission.

The transport equipment industry would thus require about 1.34 million ton of steel per year. Requirements of other industries were estimated in the same way; the grand total for industries came to about 8 million tons of rolled metal.

In other cases a different approach is necessary. The steel requirement per rupee of net investment has been estimated for different types of activities. For example, the consumption of steel is 40 tons per investment of Rs. 100,000 in railways; the corresponding figure is so low as only 5 tons in large and medium scale irrigation. The total steel requirement for a target of investment in the Fourth Plan amounting to Rs. 170,000 million can be estimated at 20 or 21 million tons.

Also, on the basis of the investment outlay for the last year of the Fourth Plan, one can estimate the steel requirement at about 5 million tons at the end of the Fourth Plan.

Adding to this the current requirement of 8 million tons for industries, the total demand would be about 13 million tons of steel in 1970-71. In the same way it has been estimated that the requirement of steel would reach 18 or 19 million tons in 1975.

Balance of electricity: It is possible in the same way to estimate the requirements of electricity from the physical targets of production for any given year. For example, the production of ferro-manganese in 1960-61 was 100,000 tons for which the electricity consumed was 500 million kwh. For a target production of 385,000 tons for ferro-manganese in 1970-1971, the requirements of electricity would be 1,952 million kwh. A similar method of calculation was used for different types of industries. Table A-3 in the Appendix gives the details. Steel and electricity are typical illustrations of the material balances which have been prepared in India for important commodities and energy for perspective planning of the economy 15 or 20 years ahead.

Perspective planning of fertilisers: The population of India is growing roughly at the rate of perhaps 9 million per year. The additional quantity of food grains required for these 9 million people would be about 1.5 million tons a year. This would add up to 22.5 million tons in the first five year period (not to speak of 60 million tons in the second five year period). At an average price of 90 dollars per ton, the cost of importing 22.5 million tons in a five year period would come to about 2,000 million dollars.

On the other hand, if imported ammonium sulphate is used, each ton on an average should increase the yield of food grains by about 2.2 tons. On this basis, roughly 10 million tons of imported ammonium sulphate would enable the domestic production of food grains being increased by about 22 million tons in a five year period. At an average price of 70 dollars per ton of fertilisers, the cost in foreign currency would be only about 700 million dollars or a third of the cost of imported food grains.

Imported foodgrains can be quickly distributed and it is possible to make necessary arrangements for such imports at short notice in the course of a year or so under normal conditions of easy availability of foodgrains in the world market. (The lack of foreign currency is the only limitation in a country like India). The import of fertilisers, however, require placing of orders a year or two or even more years in advance because the supply position is not so easy as in the case of foodgrains. Such a plan would, therefore, require taking a view of future needs two or three years ahead.

A third possibility would be to set up a new factory every year for the production of 750,000 tons of ammonium sulphate per year. At the cost of about 90 million dollars for each factory, the total expenditure would come to 450 million dollars of which, however, only 250 million dollars would be the foreign exchange requirement. The setting up of a new fertiliser factory would require at least five or six years; the process of planning must therefore start something like 10 years in advance.

Finally, it is also possible to manufacture in India machinery for the installation every year of a new fertilizer factory with capacity to produce 750,000 tons of ammonium sulphate per year. The foreign exchange requirement for this purpose would be less than 100 million dollars, to be spent once and for all. However, the installation of a plant to manufacture machinery for the production of fertilisers would take at least five or six years. When the first batch of machinery is produced, it would take another five years or so to complete the construction of a fertiliser factory. Such a plan would require a view being taken of future requirements at least 12 or 15 years in advance.

Consumer goods: In the case of consumer goods the increase in demand is estimated on the basis of the increase of income accepted as a target. Standard methods are used to calculate the elasticty of demand from information regarding expenditure (and consumption in physical terms, where possible) of a large number of commodities and services which is being collected every year by the National Sample Survey (NSS) of India. In the NSS, the design of interpenetrating net-work of sub-samples (IPNS) is always used providing at least two independent estimates of each variate. It is, therefore, possible to estimate the elasticity of demand on the basis of each sub-sample and also on the basis of the combined sample of the two sub-samples pooled together. Table A-2 in the Appendix gives estimates of percentage increases in demand over the five-year period of the Third Plan. The two independent sub-sample estimates supply useful information on the margin of uncertainty of the estimates.

In a planned economy it is not possible to allow the supply to increase with the demand without any restriction. It is necessary to increase domestic savings by restricting the consumption of non-essential or luxury goods. It is therefore necessary to impose excise and sales tax or controls on imports or on production to bring about a balance between the planned supply and the estimated demand.

Recently the method of fractile graphical analysis is being used for estimating elasticities of demand for households having different values of total per capita consumer expenditure (which is a rough indicator of the level of living). This approach has the great advantage of showing, in a very simple way, the pattern of change of the elasticity of demand with a change in the level of living. Analysis by fractile groups is particularly useful in studying the effect of excise and sales tax in balancing supply and demand.

Perspective planning of man-power: It is only with the help of skilled workers, technicians, technologists and engineers that raw materials can be converted into machinery, electricity and power which can then be used for the production of both capital and consumer goods. A rapidly increasing supply of engineers and technical personnel is essential for economic development. It is necessary to establish and broaden the base of primary and secondary education and to establish technical and scientific institutions and increase their number rapidly. The most serious difficulty is the lack of trained and experienced teachers at all levels. To build up a sound foundation for the outturn of technical personnel would take a great deal of time; it is a much more slowly maturing process than establishing heavy machine building, steel, heavy electrical or heavy chemical industries. Perspective planning is indispensable, and it is necessary to have targets twenty years or more in advance.

Scientific and technical manpower: From about 1955 a great deal of attention is being given in India to the question of technical manpower. The method used for estimating the requirements of technical personnel is simple and straightforward. Information relating to manufacturing industries for the reference period 1956 was collected as a part of the National Sample Surveys and was analysed in detail to ascertain the number of professional and technical workers (including engineers and scientists) employed in manufacturing industries. Estimates for a number of selected industries are given in Table 3 in the form of percentages of total employment (that is, number of engaged persons) in different industries. Separate figures are given in col. (2) for the proportion of professional, technical and associated workers taken together, in col. (3) for the proportion of engineers, architects and surveyors, and in col. (4) for the proportion of scientists including chemists, physicists, geologists and other physical scientists.

There are wide variations in requirements of professional and technical personnel or of engineers or scientists from one industry to another. In chemicals, and aircraft assembling and repair, the proportion of professional and technical staff is about 10 per cent. The chemical industries, naturally, require 5 per cent of scientists (no doubt, mostly chemists) and only 0.6 per cent of engineers. In contrast, aircraft assembling and repair requires a high proportion of about 5.5 per cent of engineers but practically no scientists.

TABLE 3

Technical personnel in selected industries: sample survey of manufacturing industries, 1956

industries	percentage of total employment		
	professional	engineers	scientists
(1)	(2)	(3)	(4)
1. Rice milling	0.87	0.08	0.00
2. Cotton textiles	0.90	0.12	0.51
3. Glass and glassware	0.99	0.19	0.18
4. Tea manufacturing	2.39	0.31	0.03
5. Aluminium, copper, brass: secondary products	2.49	1.58	0.05
6. Sugar	2.65	0.51	0.71
7. General engineering and electrical engineering	4.27	2.02	0.01
8. Paints and varnishes	5.44	0.31	3.47
9. Cement	5.53	0.89	1.12
10. Petroleum refining	5.56	1.55	2.40
11. Electricity generation and transmission	6.50	4.79	0.04
12. Iron and steel: primary products	6.70	2.86	0.58
13. Railway wagon manufacturing	8.46	3.02	0.21
14. Aircraft assembling and repair	9.93	5.47	0.00
15. Chemicals (including drugs)	9.99	0.62	5.06

Source: *Occupational Pattern in Manufacturing Industries, India 1956* by PITAMBAR PANT and M. VASUDEVAN with a foreword by P. C. MAHALANOBIS. Planning Commission, Government of India, 1959.

In col. (2) 'professional' stands for all professional, technical and related workers. In col. (3) 'engineers' cover architects and surveyors. In col. (4) 'scientists' stand for chemists, physicists, geologists and other physical scientists.

With any assumed target of production for any particular industry in any given year, it is possible to estimate the total number of engaged persons and hence the number of professional staff, engineers and scientists. Requirements of engineers and technical personnel were estimated in this way for purposes of perspective planning.

Expansion of technical staff: Appropriate action was taken to expand the capacity of existing scientific and technological institutions and to establish new institutions all over the country to ensure a sufficiently rapid expansion of scientific and technical personnel. The following table shows the new admissions into universities and higher educational institutions of the university standard in science and technology.

TABLE 4

Admissions into higher degree level institutions in science and technology

subject	1950-51	1960-61	1965-66	1975-76	1950-51	1960-61	1965-66	1975-76
(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)
		in thousands			as percentage of 1950-51 figures			
1. Science . .	51	116	199	264	100	227	390	518
2. Engineering .	4	14	25	70	100	350	625	1,750
3. Medicine . .	2.5	6	8	20	100	240	320	800
4. Agriculture .	2.0	5	9	15	100	250	450	750
5. Total . . .	59.5	141	241	369	100	237	405	620

On the whole the planning for scientific and technical manpower, particularly for engineering, has been quite satisfactory in India. For example, the new admissions in engineering increased from 4,000 a year in 1950-51 to 14,000 a year in

1960-61. Also, the target is about 25,000 new admissions in 1965-66, and 70,000 in 1975-76.

Outturn of engineers: The Appendix Table A-4 gives the outturn of scientists and engineers in India from 1915 to 1960. It would be seen from Table A-4, line 8 and col. (4), that the number of degree level engineers turned out between 1915 and 1947 was 14,984 in 33 years before independence. This was practically matched by a turnout of 14,385 in five years during the period of the First Plan (1951-56). The outturn increased much further to 24,166 during the five-year period of the Second Plan.

The outturn for individual years between 1951 and 1960 also shows a very rapid increase. The outturn of degree level engineers was 1,700 in 1951 which was nearly doubled in three or four years. Perspective planning of technical personnel was seriously started from 1955; the effect became visible after four years in 1959 when the outturn rose to 6,779 against 3,689 in the previous year, that is, an increase of more than three thousand in one year.

Scientific Research: Although the intake and outturn of scientists also has been increasing fairly rapidly, I am sorry to say that perspective planning of scientific research has not yet started seriously. The emergence into the modern age of any underdeveloped country would be possible only with the building of the base of science education and scientific research. Certain compelling reasons can be appreciated very easily. Natural resources are not identical everywhere; there are wide variations from one country to another. Resources available within any country can be used most effectively only through continuing applied scientific and technological research in which use is made of basic scientific knowledge to solve practical problems. It is also necessary to provide facilities for fundamental research not only for the accumulation of scientific knowledge but also to supply scientists who would be able to diagnose problems properly and identify how such problems should be handled or what kind of help should be obtained from abroad. There is also a deeper need of replacing the traditional pattern of making decisions on the basis of authority by decisions to be made increasingly on objective grounds based on scientific and rational thinking.

Perspective planning is indispensable. The need of perspective planning, especially in underdeveloped countries, may be stated very briefly in conclusion. It is necessary to increase the supply of consumer goods. To do this it is necessary to expand continually the production of capital goods. Both would require an increasing supply of engineers and technicians. Industrial and technological developments would call for a rapid expansion of applied research which, in its turn, would require a sound foundation of basic research.

The factor of time may be next considered. Factories for the production of practically any kind of consumer goods can be established in a year or two with the help of imported machinery or fuel. To develop the production of capital goods and energy would take it at least 10 or 15 years. To secure an adequate supply of engineering and technical personnel would require 20 or 25 years. To have enough scientists of ability for both applied and basic research would take at least a generation or even more. It is clear that perspective planning, looking 15 or 20 or 30 years ahead, is indispensable for all underdeveloped countries.

APPENDIX

TABLE A-1

Estimates of β and θ for major groups of manufacturing industries with 1957 and 1960-61 weights

	β		θ (thousand Rs.)	
	1957 weights	1960-61 weights	1957 weights	1960-61 weights
(1)	(2)	(3)	(4)	(5)
1. Metallurgical industries . .	0.19	0.20	178.9	172.3
2. » semi manf. .	0.47	0.45	17.1	19.0
3. Mechanical and general engineering	0.66	0.65	11.4	10.9
4. Transport equipment . .	0.45	0.45	15.4	15.3
5. Electrical equipment . . .	0.50	0.49	16.6	18.5
6. Industrial machinery (I) .	0.62	0.61	24.9	22.7
7. » » (II) .	0.47	0.43	17.4	20.1
8. Chemicals	0.35	0.32	29.1	30.3
9. Textiles	0.38	0.38	10.6	10.5
10. Rubber and leather products	0.62	0.61	14.5	14.8
11. Food industries	0.30	0.30	13.0	12.9
12. Mining industries	0.33	0.35	17.6	20.5
13. Timber and cellulose industries	0.33	0.31	11.2	12.3
14. Mining and oil industry . .	0.43	0.39	9.5	11.1
15. All industries	0.36	0.35	13.2	15.3

Note: The coefficients are obtained from detailed industry-wise information compiled by the Perspective Planning Division of the Planning Commission in collaboration with the Planning Unit of the Indian Statistical Institute.

TABLE A-2

Estimates of percentage increases in demand over the third five year plan period 1961-1966

name of item	percentage increase in demand								
	urban India			rural India			all-India		
	ss. 1	ss. 2	combined	ss. 1	ss. 2	combined	ss. 1	ss. 2	combined
(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)
1. Pulses	32	32	32	19	21	20	21	23	22
2. Vegetables	35	38	36	22	23	22	26	27	26
3. Spices	30	31	30	18	17	17	19	20	19
4. Edible oil	35	35	35	23	20	22	26	24	25
5. Sugar	38	39	38	29	28	29	31	31	31
6. Milk and milk products	46	43	44	33	40	36	37	40	38
7. Meat, fish, eggs	38	40	39	21	21	21	26	28	27
8. Fruits and nuts	46	45	45	23	25	24	32	33	32
9. Beverage and refreshments	42	41	41	28	27	27	34	34	34
10. Tobacco	36	35	36	22	19	21	25	23	24
11. Kerosene	32	33	32	19	18	19	22	22	22
12. Fuel and light	33	33	33	19	17	18	22	21	21
13. Cotton clothing (mill-made)	49	43	48	36	33	34	39	35	37
14. Washing soap	38	39	39	30	27	30	33	32	33
15. Toilets	45	41	41	22	22	22	32	30	30
16. Railway	53	49	53	46	33	35	49	40	43
17. Conveyance	53	55	53	38	28	35	44	39	43
18. Cinema	45	53	49	32	30	31	41	44	42
19. Domestic utensils	31	46	37	37	23	36	37	26	36

Note: Based on elasticities calculated in the Indian Statistical Institute, Calcutta from 10th round NSS data relating to December 1955-May 1956.

TABLE A-3

Electricity balance

consuming industry	unit	volume of production			electricity consumption in m. kwh			electricity consumption per unit of production
		1960-61 production	1965-66 capacity	1970-71 capacity	1960-61	1965-66	1970-71	
(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)
1. Iron and steel								
1 finished steel . .	m. ton	2.2	7.5	13.5	1,100	3,750	6,750	500 kwh/ton
2 pig iron	»	0.9	1.5	3.0	18	30	60	20 »
3 steel re-rolling					200	550		
2. Ferro-manganese . .	000 tons	100.0	220.0	385.0	500	1,100	1,925	5,000 »
3. Ferro-silicon . . .	»	6.0	40.0	60.0	48	320	480	8,000 »
4. Alloy steel	»	—	200.0	500.0	—	250	625	1,250 »
5. Aluminium . . .	»	18.5	125.0	250.0	370	2,500	5,000	20,000 »
6. Copper	»	8.9	22.0	50.0	27	66	150	3,000 »
7. Zinc	»	3.2	15.0	30.0	13	63	126	4,200 »
8. Coal								
1 bituminous . . .	m. ton	53.0	97.0	180.0	106	1,940	3,600	20 »

NOTES: *Source*: Perspective Planning Division paper: Demand for Electricity, India 1960-1970. The table covers all industrial uses of electricity. The norms of eletricity requirement in different industries, given in col. (9), have been used to work out the consumption of electricity in 1960-61, 1965-66 and 1970-71 given respectively in cols. (6), (7) and (8).

TABLE A-3 (continued)

Electricity balance

consuming industry	unit	volume of production			electricity consumption in m. kwh			electricity consumption per unit of production
		1960-61 production	1965-66 capacity	1970-71 capacity	1960-61	1965-66	1970-71	
(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)
2 brown	m. ton	—	6.0	12.0	—	120	240	20 kwh/ton
9. petroleum (refining) .	»	4.6	17.0	30.0	161	595	1,050	35 »
10. Fertilizers								
1 nitrogeneous, electrolytic process . .	000 tons	110.0	80.0	80.0	462	1,280	1,280	16,000 »
2 nitrogeneous, rest .	»	—	920.0	1,920.0	—	3,864	8,064	4,200 »
3 phosphatic . . .	»	55.0	500.0	1,250.0	25	225	563	450 »
11. Heavy chemicals								
1 sulfuric acid . . .	»	363.0	1,750.0	3,700.0	91	438	925	250 »
2 soda ash	»	145.0	530.0	860.0	22	80	129	150 »
3 caustic soda, chemical process . . .	»	—	50.0	90.0	—	25	45	500 »

NOTES: *Source*: Perspective Planning Division paper: Demand for Electricity, India 1960-1970. The table covers all industrial uses of electricity. The norms of electricity requirement in different industries, given in col. (9), have been used to work out the consumption of electricity in 1960-61, 1965-66 and 1970-71 given respectively in cols. (6), (7) and (8).

TABLE A-3 (continued)

Electricity balance

consuming industry	unit	volume of production			electricity consumption in m. kwh			electricity consumption per unit of production
		1960-61 production	1965-66 capacity	1970-71 capacity	1960-61	1965-66	1970-71	
(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)
4 caustic soda, electrolytic process . .	000 tons	100.0	350.0	610.0	420	1,470	2,562	4,200 kwh/ton
12. Plastics	»	10.0	85.0	250.0	1	5	15	60 »
13. Soap	»	150.0	500.0	700.0	30	100	140	200 »
14. Synthetic rubber . .	»	—	50.0	140.0	—	35	98	700 »
15. Paper and paper board	»	350.0	820.0	1,500.0	630	1,476	2,775	1,800 »
16. Newsprint and security paper	»	25.0	151.5	240.0	16	98	156	650 »

NOTES: *Source*: Perspective Planning Division paper: Demand for Electricity, India 1960-1970. The table covers all industrial uses of electricity. The norms of electricity requirement in different industries, given in col. (9), have been used to work out the consumption of electricity in 1960-61, 1965-66 and 1970-71 given respectively in cols. (6), (7) and (8).

TABLE A-3 (continued)

Electricity balance

consuming industry	unit	volume of production			electricity consumption in m. kwh			electricity consumption per unit of production
		1960-61 production	1965-66 capacity	1970-71 capacity	1960-61	1965-66	1970-71	
(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)
17. Cement . . .	m. tons	8.0	15.0	26.0	960	1,800	3,120	120 kwh/ton
18. Cotton textiles	m. metres	4,572.0	5,300.0	6,400.0	1,998	2,316	2,797	437 kwh/000 metres
19. Jute	000 tons	1,065.0	1,200.0	1,600.0	415	468	624	390 kwh/ton
20. Rayon and staple fibre								
1 rayon filament . . .	m. lbs.	47.0	140.0	250.0	147	438	781	3,125 kwh/000 lbs.
2 staple fibre .	»	47.8	75.0	120.0	40	62	99	825 »
3 chemical pulp	000 tons	—	120.0	250.0	—	60	125	500 kwt/ton
21. Woollen fabrics-yarn	m. lbs.	28.0	67.0	100.0	84	201	300	3,000 kwh/000 lbs.

NOTES: *Source*: Perspective Planning Division paper: Demand for Electricity, India 1960-1970. The table covers all industrial uses of electricity. The norms of electricity requirement in different industries, given in col. (9), have been used to work out the consumption of electricity in 1960-61, 1965-66 and 1970-71 given respectively in cols. (6), (7) and (8).

TABLE A-3 (continued)

Electricity balance

consuming industry	unit	volume of production			electricity consumption in m. kwh			electricity consumption per unit of production
		1960-61 pruduction	1965-66 capacity	1970-71 capacity	1960-61	1965-66	1970-71	
(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)
22. Silk	m. yds.	350.0	550.0	800.0	105	165	240	300 kwh/000 yards
23. Sugar . . .	m. tons	2.7	3.5	4.5	162	210	270	60 kwh/ton
24. Vegetable oil .	»	0.8	1.2	2.2	100	150	275	125 »
25. Vanaspati ghee	000 tons	330.0	550.0	730.0	73	121	161	220 »
26. Bicycles . . .	m. nos.	1.0	2.2	3.5	15	33	53	15 kwh/nos.
27. Sewing machines	000 nos.	300.0	700.0	1,200.0	18	42	72	60 kwh/nos.
28. Electric fans .	m. nos.	0.9	2.8	4.5	18	56	90	20 kwh/nos.
29. Electric lamps .	»	39.5	83.0	120.0	6	12	18	150 kwh/000 nos.
30. Matches . . .	m. gross boxes	33.0	45.0	52.0	20	27	31	600 kwh/000 gross boxes

NOTES *Source*: Perspective Planning Division paper: Demand for Electricity, India 1960-1970. The table covers all industrial uses of electricity. The norms of electricity requirement in different industries, given in col. (9), have been used to work out the consumption of electricity in 1960-61, 1965-66 and 1970-71 given respectively in cols. (6), (7) and (8).

TABLE A-3 (continued)

Electricity balance

consuming industry	unit	volume of production			electricity consumption in m. kwh			electricity consumption per unit of production
		1960-61 production	1965-66 capacity	1970-71 capacity	1960-61	1965-66	1970-71	
(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)
31. Plywood . .	m. sq. metre	15.5	27.0	45.0	14	25	42	930 kwh/000 sq. metres
32. Calcium carbide	000 tons	10.0	67.0	110.0	35	235	385	3,500 kwh/ton
33. Automobile tyres	m. nos.	1.4	3.7	7.5	150	396	803	107 kwh/nos.
34. Automobiles .	000 nos.	53.5	100.0	200.0	54	100	200	1,000 kwh/nos.
35. Coke	m. tons	5.0	15.0	25.0	125	375	625	25 kwh/ton
36. Total					8,779	27,672	47,870	

NOTES: *Source*: Perspective Planning Division paper: Demand for Electricity, India 1960-1970. The table covers all industrial uses of electricity. The norms of electricity requirement in different industries, given in col. (9), have been used to work out the consumption of electricity in 1960-61, 1965-66 and 1970-71 given respectively in cols. (6), (7) and (8).

TABLE A-4

Outturn of scientists and engineers in India

	year	number of persons graduating					
		master's degree in natural science		engineering			
		total	average per year	degree	diploma	total	average per year
(0)	(1)	(2)	(3)	(4)	(5)	(6)	(7)
1.	1915-19	832	166	568	1,703	2,271	454
2.	1920-24	917	183	771	1,902	2,673	535
3.	1925-29	1,923	385	1,619	4,322	5,941	1,188
4.	1930-34	2,784	557	2,190	5,397	7,587	1,517
5.	1935-39	2,938	588	2,901	5,331	8,232	1,646
6.	1940-44	3,378	676	3,765	6,280	10,045	2,009
7.	1945-47	2,511	837	3,170	4,538	7,708	2,569
8.	1915-47	15,283	463	14,984	29,473	44,457	1,347
9.	1948-50	2,947	982	4,691	4,623	9,314	3,105
10.	1951-55 (1st Plan)	9,062	1,812	14,385	11,629	26,014	5,203
11.	1956-61 (2nd Plan)	15,799	3,160	24,166	27,037	51,203	10,241
12.	1951	1,409	1,409	2,301	1,700	4,001	4,001
13.	1952	1,680	1,680	2,559	2,049	4,608	4,608
14.	1953	1,694	1,694	2,926	1,693	4,619	4,619
15.	1954	2,068	2,068	3,238	2,833	6,071	6,071
16.	1955	2,211	2,211	3,361	3,354	6,715	6,715
17.	1956	2,456	2,456	3,456	4,131	7,587	7,587
18.	1957	2,832	2,832	3,507	4,413	7,920	7,920
19.	1958	2,982	2,982	3,689	5,944	9,633	9,633
20.	1959	3,558	3,558	6,779	6,182	12,961	12,961
21.	1960	3,971	3,971	6,735	6,367	13,102	13,102

Note: Figures are taken from *Recent developments in the organization of science in India* by P. C. MAHALANOBIS, *Engineers in India* by Scientific and Technical Manpower Division, Planning Commission; *Education in India* by Ministry of Education, and also direct information from the Resources and Scientific Research Division of the Planning Commission.

TABLE A-5

Average per capita consumer expenditure in Rupees per month (30 days), percentage share of total consumer expenditure and limiting values of consumer expenditure by fractile groups for the 8th round of the National Sample Survey, July 1954-March 1955, all-India: Rural and Urban

fractile group (percentage)	average per capita consumer expenditure (Rs.)						percentage share						limiting values (Rs.)					
	rural			urban			rural			urban			rural			urban		
	ss. 1	ss. 2	pooled	ss. 1	ss. 2	pooled	ss. 1	ss. 2	pooled	ss. 1	ss. 2	pooled	ss. 1	ss. 2	pooled	ss. 1	ss. 2	pooled
(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)	(11)	(12)	(13)	(14)	(15)	(16)	(17)	(18)	(19)
1. lowest	—	—	—	—	—		—	—	—	—	—	—	2.19	2.20	2.19	2.76	2.43	2.43
2. 0— 10	4.21	4.64	4.48	6.20	6.68	6.54	2.75	3.03	3.01	2.96	2.46	2.65	5.48	5.89	5.70	8.14	8.52	8.30
3. 10— 20	6.25	6.56	6.42	9.30	9.96	9.59	4.07	4.41	4.09	4.27	3.48	3.90	7.01	7.39	7.18	10.22	11.24	10.82
4. 20— 30	7.72	8.24	7.99	11.62	12.83	12.06	5.05	5.57	5.33	4.56	4.72	4.85	8.26	8.93	8.67	12.68	14.02	13.29
5. 30— 40	9.26	9.52	9.37	13.75	15.19	14.28	6.11	6.19	6.18	6.73	5.34	5.70	10.15	10.18	10.17	14.65	16.91	15.30
6. 40— 50	10.91	10.89	10.90	16.00	18.56	16.94	7.18	7.43	7.27	7.05	6.86	6.78	11.74	11.75	11.74	17.42	20.17	18.50
7. 50— 60	12.61	12.65	12.63	19.00	21.62	20.11	8.27	8.35	8.35	8.29	7.82	8.01	13.73	13.70	13.73	20.91	23.79	21.73
8. 60— 70	14.82	15.12	14.94	22.68	26.82	23.86	9.51	10.00	9.74	9.98	9.71	9.64	16.23	16.69	16.42	24.45	29.65	26.08
9. 70— 80	17.72	18.54	18.17	27.20	33.52	29.52	11.78	12.23	11.95	11.65	12.16	11.86	19.72	20.83	20.22	30.92	37.20	33.48
10. 80— 90	22.42	23.80	23.04	37.56	43.54	39.00	14.78	16.18	15.53	15.43	15.78	15.61	26.79	28.03	27.55	46.71	53.61	46.65
11. 90—100	46.44	39.00	42.16	65.20	88.22	76.78	30.50	26.61	28.55	29.08	31.67	30.99	239.25	112.96	239.25	525.07	333.92	525.07
12. 0—100	14.93	14.98	14.96	22.44	27.69	25.24	100.00	100.00	100.00	100.00	100.00	100.00	239.25	112.96	239.25	525.07	333.92	525.07
13. Number of villages or blocks .	353	353	706	238	228	466	353	353	706	238	228	466	353	353	766	238	228	466
14. Number of households . .	931	938	1869	963	892	1855	931	938	1869	963	892	1855	931	938	1869	963	892	1855

Source: Indian Statistical Institute, Calcutta.

DISCUSSION

SCHNEIDER

I quite agree that, as you say, planning is necessary for underdeveloped countries. But: is it necessary because you want to have a development *in a very short time,* or because you believe that a market system would not function in India, or is it both?

MAHALANOBIS

I think that the answer is that a market system has not been found sufficient even in the U.S.A. where the largest volume of production is of free enterprise. The Anti-Trust Law was introduced about 60 years ago. The Agricultural Adjustment Act came, I believe, about 30 years ago. In the underdeveloped countries it is not possible to rely on the market system alone for improving the conditions of those who are now underfed. Also, disparities between countries are increasing because of the faster progress of industrialisation which has taken place and is continuing to take place in the advanced countries. We can not wait for another hundred years for economic development through the market system alone; we must try to introduce other methods.

Also, there is increasing interlocking of international affairs with domestic affairs. I was happy this time during my visit to the U.S.A., to see that the desegregation movement is gathering strength. The tragedy of the civil war had occurred hundred years ago; it has taken hundred years for this to happen. I see one factor in the increasing appreciation in America of the fact that the foreign policy of U.S.A., will collapse, so far as Africa or Asia is concerned, unless something is done to remove the colour bar within the U.S.A. The coloured people in the U.S.A., knowing what is happening in Africa, is also no longer content to put up with segregation. I shall go a step further; I am eager, intellectually and emotionally, that there should be increasing relaxation of the tension between U.S.A. and U.S.S.R. I am happy about the Test Ban; I do hope that it will make further progress. But I am aware that economic aid on both sides had started in the spirit of cold war. In 1946 when the Atlantic Charter was extended in the United Nations to cover the whole world, there was hope of multi-lateral or, rather, international aid, for the under-developed countries. This was turned into bi-lateral aid after the Marshal Plan. Then the Cold War started; first the military phase, and then economic aid much of which has been given as a kind of political bribery, or in political rivalry. Some deeper thought has to be given to this. I believe that world peace requires, and relaxation of tension requires, rapid transformation of the underdeveloped world. Economic cooperation would be of great help in bringing this about. Such cooperation does not in any way jeopardise the military security of the great powers. Also steel produced in India in factories set up with the help of the West German, or the British or the Russian Governments, find their way into the same piece of machinery; all aid would be necessarily cooperative in physical fact.

I have spoken of the social and political aspects of the problem of economic development because it is only through cooperative efforts of all the advanced countries that a solution can be found. The use of econometric methods must also suit the needs of the underdeveloped countries.

KOOPMANS

We have covered a great deal of ground in the last ten minutes. But my comments really go back to our earlier discussions on the use of techniques and planning. I have the impression that Professor MAHALANOBIS is using the term « sophisticated » with disapproval, and « mathematical » with approval. I would like to express my own belief that if you look at planning from the time at which it starts off, the greatest return on economic analysis is at the early « arithmetical » stage when small comparisons and computations of the kind that have been illustrated are made; and that the time for sophisticated analysis comes later whereas the payoff to it is probably substantially smaller; I think I understand the sense in which the terms « sophisticated » and « mathematical » are used in this way and take that sense over for these comments. But I do not see this as a reason to refrain from working on sophisticated methods — besides having a certain direct payoff the sophisticated methods also have the advantage of introducing rationality and objectivity in the decision process, an advantage distinct from the direct payoff itself.

I also have a more detailed comment about the fertilizer calculation, which may illustrate this general comment. There is a comparison here between different ways of supplying fertilizers. These ways differ in three respects. One is timing of availability of the fertilizer, another is rupee cost, and the third is dollar cost, or foreign exchange cost, and the timing of these. Now I would submit that to make the comparison one needs not only the dollar figures that are stated here, one also needs a shadow price on foreign exchange, and a (shadow) social rate of discount, in order to make the comparison between these three alternatives. The shadow price of foreign exchange would itself have to be determined in a wider calculation, which includes not only fertilizers but many other instances of decisions where foreign sources and domestic sources are considered as alternatives. In CHENERY's work on Southern Italy (1) the shadow interest rate came out at something like 35% per year. This is to

(1) CHENERY and KRETSCHMER, « Econometrica », 1956, pp. 365-99.

be interpreted as the real rate of interest, on loans from outside Southern Italy, that applied to the planning of the Southern Italian economy as of that time, calculated on the assumption that the production relations and planning objectives are adequately estimated in the study.

MAHALANOBIS

To reply briefly, the fertilizer question has been studied by competent foreign economists and it has been found that it would be a good choice in India to build machinery for the manufacture of fertilizers. My own position is simple. When a better solution is available I shall adopt it. But I shall not wait to find the best solution. In India, we can use 30 million tons of ammonium sulphate every year, at a very low rate of dressing and without any fear of decreasing return. We should start increasing the capacity to produce fertilizers, and not wait until an optimum solution has been established.

However, at a higher level, I completely share your views on the value of sophisticated analysis. It is my firm belief that as the structure of an economy becomes mature and stable, sophisticated analysis would become more and more capable of being used effectively. I can give one example; one of the dramatic successes of linear programming was in oil refinery because organic chemistry has stable structural relations. Sophisticated analysis would be increasingly used in the advanced countries. Also, I should think that such methods would be used more effectively in a planned economy like U.S.S.R. because factors there are more under control. On the other hand, sophisticated analysis, if it loses a sense of realism and is primarily imitative, as is likely to happen in a country like India, then it is not only useless but a menace.

I am not against sophisticated analysis in its proper setting. Even at a low level of economic development, I am in favour of introducing automatized production at the earliest opportunity and to the largest extent possible in a country like India, and at the same time to go on with traditional methods of production. The danger lies in sophisticated analysis which are unsuited to the underdeveloped countries. But we have to take the risk. In India we are going on with input/output analysis and all that kind of work in the hope that gradually this approach will become useful. In applied science the important point is to know what factors are relevant and what are their priorities. Such understanding of relative priorities exists in the more advanced countries but is difficult to acquire in an underdeveloped environment. The methods have to be suited to the stage of development.

DORFMAN

The conclusion of Professor MAHALANOBIS' fertilizer example was that the best solution for India would be to use the most indirect method of production, that is, to import the machinery for setting up a plant to manufacture machinery for fertilizer plants. This policy would require deferring the output of the different fertilizers for some ten or twelve years. If that be so, it would seem to me that the cost of importing food during this long interval to meet the annual population increase of some 5 million people should be counted as part of the cost of this roundabout method of production, because those food imports could be avoided by using some of the quicker methods, for example, by importing fertilizers or importing fertilizer plants directly.

MAHALANOBIS

I apologize. I should have stated that for 15 years our people will not be starving. In the meantime we are importing foodgrains for current consumption as necessary; we are importing fertilizers; and we are importing machinery to set up new fertilizer factories. But we are also now establishing a factory to produce machinery which, after four or five years, would enable us to set up a new ammonium sulphate factory every year. I hope the position is clear. This policy may not be suitable for a small country. I am not generalizing. But in India as I have already mentioned we have 50 million hectares of land which is sure of having enough water; we can use 25 or 30 million tons of ammonium sulphate per year at a very low rate of dressing of 25 kilos per hectare; we may be able to use much bigger quantities with advantage by increasing the dressing. To set up one heavy machine building factory at a cost of foreign exchange of not more than 100 million dollars once and for all is not taking a very big risk.

ISARD

I just have a brief question. Do the materials that you have presented have reference to a large population spread over a large area? I am sure you are aware of the thinking which emphasizes that industrialization proceeds from relatively concentrated cores of activity. To achieve major industrialization you must select a relatively small number of points at which to start. I was wondering to what extent this consideration enters into all the estimates and plans upon which you report.

FISHER

In the discussion of Professor MAHALANOBIS' paper Professor DORFMAN raised a point which I would like to emphasize and to which I do not think I really understand Professor MAHALANOBIS' answer.

Professor DORFMAN's point was that a proper analysis of the cost of the program which imports machinery to make machinery to make fertilizer must take into account the cost in foreign exchange of importing fertilizer and possibly of importing grain to feed the population during the period before the constructed fertilizer plant comes into production. This is an appropriate cost of the program; it is an opportunity cost and must be added to the cost of the program in analyzing what constitutes an optimal policy. It is not clear that Professor MAHALANOBIS has in fact done this.

MAHALANOBIS

I have only five minutes. I should again stress that it is not logical to wait an indefinite time for a true optimum solution. As the people of my country are hungry we have to import food stuff. We have been doing this for some time. As a matter of fact we have a long history of not producing a single kilo of fertilizer; we have the experience of a million and a quarter of our countrymen dying of famine in 1942-43. We are of course importing foodgrains and fertilizers for current needs. But we have to take a long view and we are also setting up fertilizer factories with imported machinery. We have spent during the last fifteen years perhaps two thousand million dollars to import foodgrains, fertilizers, and machinery to set up fertilizer factories. Looking 15 years ahead, we think it is worth spending one hundred million dollars to establish a factory to produce machinery for new fertilizer factories. This I think is the real argument.

ALLAIS

I have only two points. The first one has been already stressed by Prof. DORFMAN. I am not convinced by Prof. MAHALANOBIS's answer but I think we have not enough time to discuss this very interesting problem. My second question relates to how the equation of page 7 was derived? Did you assume that real national income is proportional to real capital for this derivation or not?

MAHALANOBIS

Please I do not quite understand. There is a footnote on page 7 which perhaps could give you the right answer.

ALLAIS

Yes, but if you could give this explanation in a few words. This point is quite important because for reasons I have already explained, it is not possible to admit that real national income is proportional to real capital.

MAHALANOBIS

The equation of the model? How we derive it? (1)

(1) The two-sector model is discussed in a paper entitled *Some observation on the process of growth of national income* in « Sankhyā, the Indian Journal of Statistics », vol. 12, Parts 4, 1953.

ECONOMETRIC ANALYSIS
AND
AGRICULTURAL AND DEVELOPMENT PLANS

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In this paper I use the term agricultural development plan as though it were synonymous with agricultural policy. I believe that this is in keeping with the usage that prevails in the world today. If one defines a plan as a statement of achievable objectives, an indication of the means or resources that will be available to achieve the objectives, and the institutional arrangements that will be used to relate the means or resources to the objectives, one can find few examples of agricultural plans. Though the details have never been published, I am reasonably confident that a plan approximately fulfilling the conditions of the above definition existed for the virgin and idle land program of the Soviet Union. In the same sense, it can be said that plans have been evolved for the development of new lands through irrigation or drainage in the United States and in many other parts of the world.

But at national levels there have probably been no development plans for agriculture that would satisfy the above definition (1). There is, of course, no denying that governments

(1) I do not believe that the agricultural components of the Soviet Union's economic plans, either the five-year plans or the current seven-year plan, meet the criteria of a plan specified in the text. In general, the

have attempted to intervene in a wide variety of ways to affect agricultural development. In the United States there has been systematic intervention in agricultural development through policies for the settlement of the frontier, the establishment of the land grant colleges and the associated experiment stations and extension services. The British Corn Laws, which have recently been reincarnated across the English Channel as the Common Agricultural Policy, could be described as an agricultural development plan in the loose sense in which the term is often used today.

The past three decades has witnessed a significant change in the scope and content of agricultural plans or policies. All of the major industrial nations have evolved numerous measures and programs in an effort to achieve a variety of objectives in agriculture and for the farm population. Many of the less developed countries have enunciated far reaching agricultural development plans. One thing that is clear from the events of the last three decades, at least to me, is that there is no magic that follows from a development plan or policy. Order and progress have not been created out of chaos. It cannot be said that today the agricultural problems confronting the world or any large part of it are any nearer solution than was true three decades ago. In fact, I believe that in most of the industrial countries the magnitude of the very difficult adjustment problems now facing agriculture is to a considerable degree the consequence of the agricultural plans and policies that have been followed.

The preliminary outline of the Study Week included the following sentence: « The development of economic theory and recent experience of various economic systems have by now

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objectives have not been achievable given the quantities of resources implied in the plans. In addition, past performance has indicated that goals for certain key input quantities, such as fertilizers, would not be achieved.

given sufficient proof that modern economies cannot be left to the free play of market forces, but must be controlled and directed both to anticipate undesirable economic situations and repercussions and to achieve certain of the community's objectives. » While I hope that I will learn what developments in economic theory and what recent experiences of various economic systems result in so striking a conclusion, for the moment I would only like to suggest the following paraphrase: « The application of economic analysis and recent experiences of various economic systems have by now given sufficient proof that modern or underdeveloped agricultures can not be subjected to agricultural development plans that impede necessary adjustments, result in exploitation of the farm population, fail to recognize the potential high returns to certain investments in agriculture, or restrict the potential gains from international specialization in production, but development plans must be consistent, efficient in the use of scarce resources, and not result in undesirable economic situations. »

In this paper I shall try to do four things. First, I shall present a simplified skeleton of some of the interrelationships that exist between agriculture and the rest of the economy as economic development occurs. Second, some of the major differences, as well as the similarities, in the agricultural problems of selected important areas of the world will be indicated. Third, a review of plans or projections that have been made in two countries — the Soviet Union and the United States — will be presented to indicate some of the major sources of errors in projections and goals. Finally, I will emphasize some of the major analytical and statistical problems that arise in making projections or creating development plans.

I. Agriculture and Economic Development (1)

A. *General Framework*

One of the important interrelationships between agriculture and the rest of the economy during the process of economic development has been called the law of declining importance of agriculture. This « law » has both very strong empirical support and a firm analytical base (2). The major industrial countries of the world, with the exception of the United Kingdom, had at the beginning of the 19th century approximately the same percentage of the labor force engaged in agriculture as is now true in the low income countries of the world. Except for the Soviet Union, all of the major industrial nations now have 20% or less of their labor force engaged in agriculture, with the United Kingdom at about 5% and the United States at 8%.

The theoretical analysis that explains the empirical decline in the relative importance of agriculture as a source of employment as economic growth occurs is relatively simple. The analysis depends upon well substantiated effects of per capita income changes upon the allocation of consumer expenditures

(1) This section and certain other parts of my paper are based in part upon my paper, *The Role of Agriculture in Economic Development*, which was presented at the 1963 Resources for the Future Forum on the Role of Natural Resources in Economic Development, in Washington, D.C. on January 28 and 29, 1963.

(2) The empirical support can be found in Colin Clark, *The Conditions of Economic Progress*, 3rd ed., 1957, Ch. 10 and Simon Kuznets, *Quantitative Aspects of the Economic Growth of Nations*: II. *Industrial Distribution of National Product and Labor Force*, « Economic Development and Cultural Change », supplement to vol. V, No. 4, July, 1957.

and upon changes in the real factor costs of agricultural products that are required if economic growth is to occur (¹).

When the level of per capita income exceeds some very low level, perhaps $50, the income elasticity of demand for food is less than unity. Since the income elasticity of demand for all goods and services is unity, the income elasticity of demand for nonfarm goods and services exceeds unity. An increase in real per capita income, ignoring for the moment the source of the increase, will result in a larger increase in the demand for nonfarm than for farm products. Thus the relative output of farm products will decline as a share of the total output.

However, the assumptions made are not sufficient to show that the share of the labor force engaged in agriculture will decline. If the growth in the net marginal product of labor engaged in nonagriculture is enough greater than the increase in the net marginal product of labor engaged in agriculture, the proportion of the labor force engaged in agriculture could remain unchanged. As a rough approximation, if there is no change in population, if the ratio of the change in the marginal product of labor in agriculture to the change in marginal product of nonfarm labor is equal to the ratio of the income elasticies of demand for food and nonfood products, the sectorial distribution of employment will remain unchanged. However, this rough approximation will hold only if there are barriers to the movement of labor from agriculture into nonagricultural pursuits. The larger increase in the marginal product of labor in the nonagricultural sector than in the farm sector means that the return to labor in agriculture will fall in a relative sense. Thus labor will move from agriculture into the nonagricultural sector in response to a change in the earnings

(¹) HERBERT SIMON, *Effects of Increased Productivity upon the Ratio of Urban to Rural Population*, « Econometrica », XV, No. 1 (January, 1947), 31-42.
F.H. GRUEN, *Agriculture and Technical Change*, « Journal of Farm Economics », XLIII, No. 4 (November 1961).

differential and the percentage of the labor force engaged in agriculture will decline. Obviously, if the relative increase in the marginal product in the nonfarm sector is large enough, a rise in the relative price of farm products could offset the decline in the earnings to farm labor and maintain the same distribution of employment. However, this consequence will almost certainly place a severe restraint upon the rate of growth of per capita income. It means that virtually all of the increased productivity must come in a sector of the economy that employs a very small fraction of total labor.

If population increase is introduced into the picture, the relative growth of demand will not be a direct function of relative income elasticities. The relative growth will depend upon the annual population growth plus the change in per capita income weighted by the income elasticities. But the growth of population means that a significant part of the increase in agricultural output is required to maintain per capita consumption of food. Unless additional land of quality equal to that already under cultivation is available, a change in methods of production will be required to offset the effect of diminishing returns as the amount of labor applied to each unit of land increases. Unless such a change in methods of production occurs, the real cost of food will increase and resources that could be used to increase nonfarm output will be shifted to agriculture and the rate of growth of total output will decline.

Significant rates of economic growth can occur only if there are increases in productivity in a sector that employs the major fraction of all of the labor of an economy. Fortunately, increases in resource productivity in agriculture are possible and the same set of forces that result in higher productivity in nonfarm sectors appears to have been roughly as important in agriculture as in the rest of the economy.

The factors that make it possible to increase agricultural output per capita in the face of increased population, which will be discussed in some detail below, are the factors that contribute to the decline in the relative importance of agriculture. These factors result in an increase in output per unit of input in agriculture; combined with the lower income elasticity of demand, subject to the conditions noted above, relative labor employment in agriculture will decline.

As real per capita income increases, the rate of the transfer of labour out of agriculture required to maintain an equilibrium in returns to farm and nonfarm labor will almost certainly increase. One reason is that the disparity between the income elasticities of demand for farm and nonfarm goods increases. A second reason is that the change in ratio of output per unit of input in agriculture is more likely to approach the change in the ratio in the nonfarm sector as agriculture becomes more commercialized and the quantity and variety of producer goods available to agriculture increase.

There is no reason to believe that there are any forces to halt the decline in the share of labor employed in agriculture. There is only one industrial country, the United Kingdom, for which there is any evidence that the percentage of total employment in agriculture has been approximately stabilized for any period of time. Since the mid-thirties approximately 5% of the total labor force has been engaged in agriculture. But during that period, the government has followed a conscious policy of inducing an increase in agricultural output. It is almost certain that in the absence of that policy relative farm employment would have declined.

B. *Economic Development and the Demand for Agricultural Products*

It is unfortunate but apparently true that the growth of demand for agricultural products is likely to be greater during the early stages of economic development than during the later stages. Two factors that affect the demand for agricultural products are responsible for this — population growth and the income elasticity of demand. One of the well known effects of the early stage of economic growth is a decline in mortality. The improvement of sanitation, the availability of medicines and medical facilities, the improvement of the quantity and quality of the food supply, the reduction in famines due to improved transportation, and the control of certain endemic diseases have a significant and short run effect upon death rates, especially of infants and children. There is almost nothing that occurs during the early stages of economic development that has an effect on fertility. Thus there is a very strong probability that the rate of population growth will be higher during the early stages than in the later stages of economic development and that at a given time higher for low income countries than for high income countries (1).

It has been noted above that the income elasticity of demand for food is higher at low than at high income levels. Thus if a high and low income country have the same growth rate of per capita income, the income effect on demand will be substantially greater for the latter than for the former. The two effects are illustrated by Table 1, which presents certain projections made by FAO for a period of roughly a decade.

The estimates of population increase are relevant to the comments made above. The three low income areas have

(1) This statement is based on the assumption that the growth in food output is enough to sustain the growth in population. If the conditions implied by a Malthusian model of economic growth apply, the growth of population will depend upon the growth of food output.

The statements in the text about the relationship between population growth and income levels are admittedly very crude and approximate generalizations. Obviously there are many factors that influence population growth. The difference in population growth rates in Western Europe and the United States can not be explained by the difference in the level of per capita income. An interesting article by IRMA ADELMAN (*An Econometric Analysis of Population Growth*, « American Economic Review », vol. LIII, No. 3, June 1963, 314-39) provides some support for the view expressed in the text.

projected annual rates of population growth ranging from 2.3 to 2.7%. The two high income areas have lower rates, namely 0.7 and 1.8%. The difference in estimated income elasticities are of the general order that would be expected, except that the elasticity for the EEC appears to be somewhat high, but is apparently explained by the high income elasticity of demand for meat.

TABLE I

Projected Annual Growth of the Potential Demand for Food Selected Areas, 1957-59 *through* 1969-71 (1)

	Asia and Far East(2)	Near East and Africa(3)	Latin America(4)	E.E.C.(5)	Nort America(6)
Basic assumptions					
Population (7) . .	2.3	2.5	2.7	0.7	1.8
GNP/capita (7) .	1.3	1.5	2.0	3.9	1.3
Income elasticity .	0.9	0.7	0.6	0.5	0.16
Potential increase (7) in					
Total demand . .	3.4	3.5	3.7	2.1	1.9
Per capita demand	1.0	1.0	1.0	1.4	0.16
GNP/capita (8) . . (1957-59)	$165	$260	$491	$1285	$2190

(1) Food and Agriculture Organization, *Agricultural Commodities - Projections for 1970,* p. A-2.
(2) Excludes Japan and Mainland China.
(3) Excludes South Africa.
(4) Excludes Argentina and Uruguay; includes Mexico and Central America.
(5) European Economic Community.
(6) Canada and the United States.
(7) Per cent increase per year; compound rate. Low income growth assumption used.
(8) Converted into U.S. $ at 1955 prices.

The increases in total demand for food are projected at annual rates ranging from 3.4 to 3.7% for the low income countries and 1.9 to 2.1% for the high income areas. Thus the expected growth of demand for food in the low income areas is approximately 75% greater than for the industrial areas. In the low income areas, despite the high income elasticities and modest increases in per capita income, most of the increase in demand results from population growth. If Latin America were to have a 3% annual increase in GNP per capita, the demand growth for food would be in excess of 4% annually. This would require a doubling of food output in about 18 years.

C. *Economic Development and the Supply of Agricultural Products*

In the discussion of the general frame-work of the relationship between agriculture and economic development the discussion of changes in supply were largely in terms of output changes that had to occur if economic development were to occur. It is time now to turn to a discussion of the major factors that lead to output increases. The following classification includes nothing that is new, but is useful for organizing our discussion. Increased agricultural output occurs as a result of one or more of the following: 1) increased use of inputs; 2) improving the quality of inputs; 3) increased knowledge or a change in the production function, and 4) a change in incentives for farm operators and their families.

If all of the increased output must come as a result of increased labor and land, it is unlikely that agriculture will be able to make any significant contribution to economic development (1). If this is the case, the growth of the agricul-

(1) This observation may well have been correct for the United States in the period before 1860. Given the large quantity of land that was available for settlement, it was possible to expand output of farm products without changing the relative combination of land to labor. Admittedly scanty evidence for the period from 1820 through 1860 indicates the follow-

tural population in a low income or underdeveloped economy will have to occur at approximately the same rate as the growth of total population unless per capita food consumption is to decline. Labor cannot be released to other sectors of the economy for their development and real factor costs of food would not decrease. In fact, unless additional land can be brought under cultivation or land developed through irrigation or reclamation at the same rate as the population grows, average output per worker in agriculture may decline and real factor costs of food may increase.

There are circumstances in which an unchanged production function is consistent with increased output and lower marginal costs of production. If the marginal product of labor is substantially higher in certain nonfarm pursuits than on the farm, the availability of certain manufactured inputs that may be used in agriculture may permit an increase in output per unit of farm labor and a transfer of labor from agriculture to the nonagricultural sector. This is a possible explanation of the impact of a new and cheaper source of fertilizer (the partial replacement of manure, fish, fish meal, and cottonseed meal by chemical fertilizers in Japan), the replacement of feed by petroleum products to produce power, or the transfer of certain functions performed on farms to firms that can specialize to a greater extent (churning butter, tanning hides, retting flax, or butchering livestock).

There is evidence that for the half century from 1880 to 1930 in the United States the ratio of total output to input did not change significantly, while there was an increase in the output per unit of labor and per unit of land (real estate).

ing: Farm employment increased from 2.2 to 6.6 million; farm employment as a percentage of total population changed only from 21.5 to 19.7%. Estimated gross output per farm worker increased by only 11% in the 40-year period. There are some indications, admittedly crude, that national per capita income increased only moderately during the four decades.

The increases in the two average products were apparently due to increased purchases of nonfarm products, which were apparently cheaper than the farm inputs they replaced. While farm labor increased absolutely over the period, farm labor declined as a percentage of the total labor force from 49 to 21%.

The United States situation as of 1880 is one that is not likely to be duplicated in any of the current low income areas of the world. U.S. per capita income was already at a relatively high level (about $400 in 1929 prices) and the income elasticity of demand for farm products was probably 0.5 or lower. About a fifth of farm output was exported, thus domestic consumption could increase more than domestic output.

TABLE 2

Indexes of Output, Inputs, and Productivity, United States Agriculture, 1880-1930 (1)

(1847-49 = 100)

Indexes	1880	1890	1900	1910	1920	1930
Farm Output	37	43	56	61	70	72
Production Inputs	53	63	73	82	93	97
Farm Labor	100	116	127	135	143	137
Farm real estate	56	66	78	93	97	96
All other	18	24	31	39	53	62
Productivity	70	68	77	74	75	74
Output/labor	37	37	44	45	48	52
Output/real estate	66	63	71	66	72	75

(1) Source: RALPH A. LOOMIS and GLEN T. BARTON, *Productivity of Agriculture, United States, 1870-1958*, USDA, Tech. Bul. No. 1238 (April. 1961), pp. 57, 58, and 60-61.

But when per capita incomes are low and the income elasticity of demand for food is almost unity, agricultural employment must increase absolutely and remain a relatively stable percentage of total employment unless technological change occurs.

It is quite likely that improved quality of inputs and a change in the production function (technological change) usually occur together. In an underdeveloped economy the most important input — as measured by the share of the total product paid to it — is labor. Fortunately, labor is an input whose quality can be changed by well known procedures. Unfortunately, these procedures involve the expenditure of significant amounts of scarce resources. But the history of Japan from 1870 through 1910 indicates that a poor country, with very limited natural resources, can organize its resources to provide for nearly universal education for the farm population, for agricultural research, and for education of the adult farm population in new and improved methods of production.

There is now rather general agreement among competent scientists that the technological possibilities for increasing food output in the low income areas are very substantial. Many of the low income areas — Africa, Latin America, and Southeast Asia — have room for substantial expansion of the cultivated areas. But some of the most densely populated areas — India and China, for example — are not in this fortunate position. Again reference may be had to the Japanese experience. Japan at the time substantial economic progress began had little possibility of increasing the sown area. In the half century from 1881-90 to 1931-40 the cultivated area of six major crops in Japan increased by only 18% (1). However, yields increased by 66% and production by 95%.

The excellent *Report on India's Food Crisis and Steps to Meet It* prepared by the Ford Foundation team clearly indicates that it is technically possible to substantially increase food production in India (2). Once of the major conclusions of the

(1) BRUCE F. JOHNSON, *Agricultural Development and Economic Transformation: Japan, Taiwan and Denmark* ,Mimeo., 1960, p. 28.

(2) Published by the Government of India, Ministry of Food and Agriculture and Ministry of Community Development and Cooperation, April 1959. The chairman of the team was Mr. Sherman Johnson of the United States Department of Agriculture.

study should be noted, however. In the opinion of the team, major increases in output can be achieved only as a result of the simultaneous introduction of a variety of improvements (3). For example, a substantially higher yield for a crop would require new varieties, more fertilizer, heavier rates of seeding or planting, measures for controlling insects and diseases, and changed methods of cultivation. Any one of these changes may be largely ineffective or may even have a negative influence in specific cases.

That it is technologically possible to increase food output is, of course, only a necessary condition for a solution of the agricultural problems of the low income areas. Much must be done before the technological possibilities can be translated into food available for an increasing population with a rising per capita level of consumption. One of the major problems in making the projections necessary for a development plan is that of estimating when and to what degree the political and economic circumstances will result in the realization of the technological possibilities.

The last of the four factors that influence the level of agricultural output is the nature of the incentives available to farm families. The incentive structure has two major elements — the terms of trade between agriculture and the rest of the economy and the relationship between effort and reward as affected by institutional factors as codified in the tenure system and the structure of the farm organization, such as individual proprietorship, collective or cooperative farms.

In a growing economy, starting from a low income level, savings in or from agriculture are an important source of investment funds for the entire economy. The extent, and the method

(3) *Ibid.*, p. 18. « A few improved practices can be effective if adopted singly, but the full benefit from most improvements can be obtained *only* if they are adopted in combinations suitable for specific soil and climatic conditions. »

of extracting such savings from agriculture, through taxes, rents, low prices or forced deliveries at nominal prices, is clearly a factor in the incentives available to farm workers. In my opinion, the relatively poor output performance of agriculture in the Soviet Union from 1928 through 1953 was due to ineffective and insufficient incentives imposed by a combination of the price structure, forced deliveries and institutional arrangements. On the other hand, during the late nineteenth century Japan apparently transferred significant amounts of savings from agriculture to the rest of the economy and still achieved rapid modernization of agriculture and output growth. Thus it is probable that not only the total amount of savings extracted from agriculture, but the method of extraction is important in determining output responses.

A part of the support for land reform in underdeveloped areas is the hope that reform will lead to an improvement in the incentive structure and thus to increased investment in agriculture and greater interest by the farm operator in increasing output. Some land reforms seem to have achieved this result (the short lived reform in the Soviet Union after World War I and the post World War II reforms in Japan and Taiwan), but many others appear to have failed to have any significant effect.

The increase in the ratio of output to input, which is a measure of the effect of the change in quality of inputs as well as of technological change, makes it possible to transfer labor from the farm to the nonfarm sector. Rapid economic growth requires such a transfer. However, rapid economic growth does not require an absolute decline in farm population or farm employment. If farm employment is to be stable in an absolute sense, the annual change in the ratio of output to input in agriculture in low income countries must be a very large change if population and income growth per capita are of the order of 2% per annum. In this situation the increase in demand for farm products (assuming little international trade) will be about 3.5% per annum. Even if half of the increased output is due to increased quantities of nonlabor inputs, the ratio of output to input must increase by about 1.5% each year. This implies a doubling of the output-input ratio in about 45 years; in the last 50 years the output-input ratio has increased by about 75% in the United States.

I should hasten to add that it is not essential for the absolute level of farm employment to remain stable during the early period of industrialization. The usual pattern has been for the farm labor force to increase absolutely for several decades after per capita incomes have started to increase. In fact, if an economy starts from a position of, say, 80% of its labor force engaged in agriculture and if total population is increasing by 2% per annum, it would be nearly impossible for nonfarm employment to absorb all of the increase in labor force. Under these assumptions, the nonfarm labor force would have to grow at a 10% annual rate or double about every 7 years (1).

In this section I have tried to indicate briefly some of the interrelationships between the supply of agricultural products and the process of economic development. It has been indicated that an improvement in the output-input ratio is required before rising per capita incomes can be achieved in a low income area. It has also been indicated that if population growth is moderate or high and the income elasticity of demand is high that the increase in the output-input ratio must be a significant one if agriculture is not to act as a restraint on economic growth. It also was indicated that while farm employment must decline as a share of total employment, it may increase absolutely during the early period of industrialization.

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(1) Such a growth cannot be said to be impossible; between 1926 and 1934 the annual increase in nonfarm employment was about 9% in the Soviet Union.

II. Major Agricultural Problems

Governments have various types of concern about the agriculture of their respective nations. In many of the low income countries there is a concern about the level of agricultural output relative to what would be required to achieve satisfactory nutritional levels for the present population and to meet the demands that will come from a growing population. In most of the high income countries this concern arises with respect to the level of returns to resources engaged in agriculture. In the Soviet Union and other bloc countries in Eastern Europe the basic agricultural problem is to achieve a rapid increase in agricultural output, especially marketed output, to provide a growing urban population with a diet that compares more favorably with that available in the Western countries and to do this without requiring a significant diversion from the industrial and military sectors.

Table 3, prepared by the U.S. Department of Agriculture, summarizes the available data on changes in total and per capita production of agricultural products for the past quarter century. In general, the Southern Area includes most of the underdeveloped areas of the world, while the Northern Area includes the medium and high income countries. With respect to total production, the increases are of the same general order of magnitude. In fact, if adjustment were made for the very unsatisfactory weather conditions in North America during 1935-39 and the very good weather conditions from 1958 through 1960 in the same area, total output probably increased more in the underdeveloped areas. It is true, of course, that within each of the two areas and within each of the regions there were significant differences from country to country in the growth of output. However, the point that I wish to make here is that the types of agricultural problems faced by the

TABLE 3

Indices of world agricultural production: Total and per capita, by region, average 1935-39 *and annual* 1958-59 *to* 1960-61 (1)

(Average 1952-53 to 1954-55 = 100)

Region or country	Total production: Average 1935-39	Total production: Average 1958-59	Total production: Average 1959-60 (2)	Total production: Average 1960-61 (3)	Total production: Average annual percent change 1935-39 to 1960-61	Total production: Average annual percent change 1952-54 to 1960-61	Per capita production: Average 1935-39	Per capita production: Average 1958-59	Per capita production: Average 1959-60 (2)	Per capita production: Average 1960-61 (3)	Per capita production: Average annual percent change 1935-39 to 1960-61	Per capita production: Average annual percent change 1952-54 to 1960-61
					Percent						Percent	
Southern Area												
Latin America	72	121	123	124	3.1	3.4	103	107	106	104	0.04	0.6
Africa and West Asia	77	117	118	121	2.5	3.0	100	106	105	105	0.2	0.7
Far East, less Japan (4)	89	117	119	119	1.5	2.7	111	104	106	105	—0.3	0.7
Communist Asia	96	120	115	117	1.0	2.4	112	109	102	102	—0.4	0.3
Total	85	118	118	120	1.7	2.9	106	106	105	104	—0.09	0.6
Northern Area												
Western Europe	81	110	112	115	1.8	2.1	92	106	107	109	0.8	1.3
Eastern Europe (5)	108	132	130	131	.9	4.4	106	123	120	119	0.5	2.7
United States and Canada	69	113	114	117	3.0	2.4	87	104	103	104	0.9	0.6
Japan	83	132	140	145	3.3	6.4	102	125	131	134	1.3	4.9
Australia and New Zealand	76	120	121	124	2.7	3.4	100	107	106	107	0.3	1.0
Total	84	118	119	122	1.9	3.1	93	111	110	111	0.8	1.6
World total	85	118	119	121	1.8	3.0	101	108	107	107	0.3	1.0

(1) Value of production at constant prices. Revised. Crops included in the index are harvested mainly between July 1 of the first year shown and June of the following year. For a few crops and most livestook production, estimates are for the calendar year of the year shown. - (2) Preliminary. - (3) Estimated. - (4) Includes Pacific Islands. (5) Includes Soviet Union.

Source: U. S. Department of Agriculture, *The World Food Budget, 1962 and 1966*, Foreign Agricultural Economic Report No. 4, Revised January 1962, p. 76.

three major areas do not seem to be a function of differences in output growth, but in the differences in demand changes compared to changes in output. On a per capita basis, the Southern Area has apparently had a much slower output growth than the rest of the world.

The remarkable similarity in the growth of output in the various areas of the world means that the underlying causes of their quite different problems must be somewhere else. In the case of the low income countries, the source of the problem is not difficult to detect. If there is rapid economic growth, the demand for farm products will increase at an annual rate of 3.5 to 4%. This increase may be compared to an estimated annual growth rate of output of about 1.5 to 3.0% for the past quarter century and 2.7 to 3.4% for the past decade.

The nature of the problem in Eastern Europe is similar to areas with rapidly growing demand for food, though there is an important distinguishing element. In Eastern Europe the growth in the demand for food will be substantially greater than in Western Europe or the United States. While the growth of population may be expected to be somewhat less than in the United States, it will probably be much more rapid than in Western Europe (1). The income elasticity of demand for food in Eastern Europe — based on scanty information from the Soviet Union — is relatively high; the income elasticity is probably about 0.75 and almost certainly greater than 0.5. Thus rising consumer incomes, perhaps at an annual per capita rate of about 3%, combined with an annual population growth rate of 1.4% would result in a growth of demand for food of 3 to 3.5% annually.

But the growth in demand is here estimated on the assumption that there now exists an equilibrium between the demand

(1) U.S. Department of Agriculture, *The World Food Budget, 1962 and 1966*, Foreign Agricultural Economic Report No. 4, Revised, January 1962, p. 72.

and supply for food. In the Soviet Union, for most food products there is excess demand at the fixed prices in the state store. Generally the prices in the free market are substantially higher than those in the state stores, for several important products 50 to 100% more. The increase in the retail price of meat by 30% and of butter by 25% on June 1, 1962 without eliminating all of the excess demand (reports of queues and bare shelves persist) indicates that some of the food markets are substantially out of equilibrium. While it is true that state food store prices are generally substantially below prices in the free market, the state food store prices are not low by comparison with Western Europe and the United States nor are they low compared to the income of the population. Approximately 50% of consumer income is expended for food, of which about 90% is purchased in state stores at controlled prices.

In Western Europe and the United States the « agricultural problem », as viewed by most governments, is to maintain a satisfactory level of farm incomes. Generally speaking, the growth of demand is less than the potential increase in production. Rapid adoption of new methods of production by farmers means that the demand for farm labor at current returns to labor is declining absolutely. When one adds to this the fact that alternative earnings in the rest of the economy are increasing, substantial reductions in labor inputs are required in most of the countries. While some of the countries have had programs designed to facilitate the transfer of labor out of agriculture, most have tried to solve the problem of declining demand for farm labor by various forms of subsidies and price maintenance or increasing measures.

The agricultural programs of the higher income countries have certain direct implications to the agricultural development plans or policies of the low income countries. Most of the low income countries depend upon a limited number of agricultural products for the bulk of their export earnings. If the major industrial countries follow policies that result in expanding agricultural output, the export earnings of the low income countries are adversely affected. It is sometimes argued that the agricultural policies and the related import and export policies of the industrial nations have little effect on tropical products and, since most underdeveloped areas are tropical, very little effect on the exports of underdeveloped areas. It is true that most of the restrictive measures undertaken by industrial countries affect cereals, which are grown mainly in temperate zones and in developed nations. But sugar is heavily protected in most industrial countries and cane sugar is a tropical product. And the grain policies of industrial countries do have an impact upon the market for rice, which is primarily a tropical product. Underdeveloped areas are a major source of vegetable fats and oils, which are subject to import duties in some industrial countries. Sugar, rice, fats and oils are major sources of foreign exchange for many underdeveloped areas; the ability of such countries to import capital equipment and other requirements for economic growth are affected directly by the agricultural and trade policies of the industrial nations.

In summary and stated in broad terms, development plans or policies for agriculture are primarily concerned with achieving a rate of growth of output that is approximately the same as the rate of growth of demand. In the underdeveloped areas of the world and in Eastern Europe, the plans or policies must concentrate upon measures that will increase the agricultural output growth rate. Obviously this must be achieved with an expenditure of resources that permits attaining other important objectives. In the United States the policies should create a situation in which output growth is no greater than the growth of demand and at the same time achieve a level of return to resources engaged in agriculture approximately equal to the returns received by comparable resources engaged elsewhere. In Western Europe the substantial imports of food give the region the choice of expanding food output to supply the regional demand or of continuing to import substantial quantities of food. The first alternative appears to imply a substantial increase in the cost of food, while the second alternative has important implications to a number of exporters of farm products. The first alternative implies a protectionist policy while the second alternative would permit the region to realize significant gains from international trade and specialization.

III. A Review of Some Agricultural Projections

A governmental plan or policy depends upon projections of future events. Plans by private firms and individuals obviously require projections of the same general kind, though the degree of detail required may be substantially less. There are, of course, significant differences in some of the effects of errors in projections or forecasts when made by governments and private individuals. For one thing, errors made by private individuals may be offsetting. For another thing, errors made by private individuals may bring into play forces to correct the error, such as a decline or increase in market price, while a government price policy, subject to rather more slowly functioning political processes, may compound the consequences of projection errors. This will happen (and has) if a farm product price is established at a high level which resnlts in attracting additional resources into the production of the product and the bidding up of the price of certain resources. In order to avoid economic distress to resources engaged in the production of the product, prices may not only be maintained at the previous level but may be increased in order to provide a satisfactory income to the overexpanded sector of the economy.

In this part of my paper I shall review agricultural projections for the United States and agricultural goals for the Soviet

Union (1). For the United States we have projections made for both consumption and production. Only one aspect of the production projections will be reviewed, namely crop yields. It is assumed that the goals for the Soviet Union represent projections of attainable output. For both the United States and the Soviet Union the projections will be related to actual experience.

A. *Projections for the United States*

1. *Consumption of agricultural products*

I shall describe and comment upon only two of the numerous projections of consumption of agricultural projects. One projection was made in 1948 by the U.S. Department of Agriculture and was for the average of the period 1955-65. I have assumed that the data for the year 1960 represent the average for that period. The other projection was made by REX F. DALY of the U.S. Department of Agriculture and was published in 1956 and the projections were for 1960; this study also included projections for 1975 but these are not considered here.

Basically the two studies projected the consumption of agricultural products on the basis of a model in which population and per capita income were projected for the specified period. Then the consumption of agricultural commodities was estimated from studies of income elasticities of demand and other information concerning trends in relative demands. In order to eliminate the effects of errors in population projections, all data on consumption are here presented on a per capita basis.

In Table 4 I present certain of the underlying data on po-

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(1) I wish to acknowledge my obligation to Professor JAMES BONNEN of Michigan State University who brought together much of the material on the U.S. projections.

TABLE 4

Population, income, output, employment and per capita food consumption projects for 1960

Item	Unit or Base	Projections for 1960: 1948 Study(1)	Projections for 1960: 1956 Study(2)	Actual 1960
Population	Million	158	178.6	179.9
Labor Force	Million	66	72.0	72.8
Employment	Million	62	68.5	68.9
Unemployment	Million	4	3.5	3.9
GNP in 1947 prices	Billion $	292.9	—	—
GNP in 1953 prices	Billion $	—	430	—
GNP in 1960 prices	Billion $	405.8	501	503
GNP per capita, 1960 prices	$	2,568	2,805	2,795
Personal Disposable Income (1960 prices)	Billion $	—	349	354
Per capita (1960 prices)	$	—	1,959	1,969
Per capita food consumption	1935-39 = 100	121.0	114.7	113.6

(1) Long-Range Agricultural Policy: A Study of Selected Trends and Factors Relation to the Long-Range Prospects for Agriculture, Committee on Agriculture, U. S. House of Representatives, 80th Congress, 2nd Session, Washington, D.C., March, 1948, pp. 28 and 34. Projections were for average of 1955-65 and for high employment conditions.

(2) REX F. DALY, *The Long-Run Demand for Farm Products*, « Agricultural Economics Research », Vol. VIII, No. 3, July 1956, pp. 7.

pulation, employment, GNP, and per capita income for the projections with comparisons with actual data for 1960.

The 1948 study underestimated 1960 population by 22 million or 12%. The relative error in the estimate of the labor force was much smaller, though still very substantial since all persons in the labor force in 1960 were alive in 1947 (the last year of available data for the 1948 study). Gross national product was underestimated by about 20%, though GNP per capita was in error by only 8%.

The 1956 study projections for 1960 were remarkably accurate for the underlying data on population, labor force, total and per capita gross national product. This study covered a period of approximately seven years since few, if any, data were used for any year after 1953.

Projections of per capita consumption of a number of agricultural commodities are compared with the actual consumption in 1960 in Table 5. The 1948 study had quite accurate projections for total meats, potatoes, and sweetpotatoes, and fats and oils. The difference between projection and actual was substantial for lard, poultry, eggs, total milk, wheat, and cotton. The differences between the 1956 study projections and actuals were not as great as in the former study, but were still substantial for certain individual commodities — beef, veal, turkeys, eggs, milk, and cotton.

Table 6 presents comparisons between projections and actual yields. The 1948 study projects yields for 1965; the 1956 study was only for demand. In Table 6 I have included projections from several other studies in addition to the 1948 study. These include projections for 1950 which were published in 1945, a U.S. Department of Agriculture projection for 1965 published in 1961, the Paley Commission projections for 1975, and U.S. Department of Agriculture projections for 1975. Actual yields are given in the table for two three-year periods - 1954-56 and 1960-62.

The 1965 yield projections from the 1948 study were ex-

TABLE 5

Projected and actual per capita food consumption, United States, 1960

	Projections for 1960 1948 Study(1) (pounds)	1956 Study(2)	Actual 1960
Meats			
Beef		74.0	85.0
Veal		9.5	6.2
Lamb and mutton		4.5	4.8
Pork (excl. lard)		68.0	65.2
Total	155-160	156	161.4
Lard	12		7.7
Poultry			
Chicken		24.0	28.2
Turkey		4.5	6.2
Total	22 (3)	28.5	34.4
Eggs (number)	360	380	324
Total Milk Equivalent	885	698	653
Fluid milk and cream	463	395	351
Condensed and evaporated			
Cheese	8	7.5	8.4
Butter	15		7.5
Potatoes and Sweetpotatoes	105-110	107	108.3
Fats and Oils	46	44.7	45.4

(1) See footnote 1, Table 4, p. 32.
(2) See fotnote 2, Table 4, pp. 10 and 12.
(3) Adjusted from dressed weight to ready to cook.

TABLE 6

Projected and actual acre yields of grains, U.S.: 1950, 1954-56, 1960-62, 1965 *and* 1975

	1950 Proj.(1)	1950 Actual	1954-56 Actual	1960-62 Actual	1965 Proj.(2)	1965 Proj.(3)	1975 Proj.(4) « B »	1975 Proj.(4) « A »	1975 Proj.(5) Attainable	1975 Proj.(5) Maximum
Wheat (bu.)	13.3	—	19.4	25.1	14.6	23	19.8	25.0	24	27
Corn (bu.)	33.8	—	41.5	60.1	35.8	51	55.0	80.0	53	61
Oats (bu.)	—	—	25.9	43.6	36.5	39	42.0	60.0	42	52
Barley (bu.)	—	—	28.3	32.0	—	32	—	—	35	42
Gr. sorghum (bu.)	—	—	20.4	42.5	—	32	26.6	32.0	35	42
Soybeans (bu.)	22.5	21.7	20.6	24.4	21.9	24	26.3	32.9	26	30
Potatoes (cwt.)	88.8	152.6	164	196	114	—	164.4	227.4	208	276
Cotton (lbs.)	261	269	389	446	315	480	401.8	500.0	495	616
Tobacco (lbs.)	1,023	1,269	1,469	1,765	1,200	—	1,378	1,566	1,422	1,541

(1) U. S. Dept. of Agriculture, *What Peace Can Mean to American Farmers: Postwar Agriculture and Employment*, Misc. Pub. 562, May, 1945.

(2) *Long-Range Agricultural Policy: A Study of Selected Trends and Factors Relating to the Long-Range Prospects for Agriculture*, Committee on Agriculture, U. S. House of Representatives, 80tb Congress, 2nd Session, Washington, D.C., Marcb, 1948, p. 46.

(3) U. S. Department of Agriculture, *Farm Production Trends, Prospects, and Programs*, Agri. Inf. Bul. No. 239, May, 1961, p. 92.

(4) *Resources for Freedom: A Report to the President's Materials Policy Commission* (The Paley Commission), June, 1952, Vol. V, p. 66. The « A » yield estimate « based on the assumption that all commercial agriculture of the U.S. is organized and managed to make full use of all available technology where such use would add more to farm receipts than to expenses. » The « B » estimate was formulated on the basis of « a projection to 1975 of the yield likely to come about from such application of available techniques as can reasonably be expected on the basis of past experience. »

(5) *U.S. Department of Agriculture, Our Farm Production Potential, 1975*, Agri. Inf. Bul. No. 233, Sept., 1960, p. 6. « The *economic maximum yield* is based on full, efficient economic application of presently known technology under assumed economic conditions. *Economic attainable* yields are yields that would be expected, by 1975, from actual application by farmers of presently known technology. » (*Ibid.*, p. 3).

ceeded by 1960-62 yields in every case where comparison is possible; the percentage excess ranged from 19 to 68 [1]. It may be noted that the 1965 projections from a study published in 1961 were equalled or exceeded in six out of seven cases by 1960-62 yields.

At the time they were made, the projections of the Paley Commission were criticized for being unrealistically optimistic. It may be noted that the lower of the two estimates (the « B » projections), except for soybeans, have already been exceeded by the 1960-62 yields. The last set of projections included in the table were published in 1961. Most of the « attainable » projections have been exceeded or equalled.

It should be noted that the yield projections have represented what might be called judgment estimates. In some instances past trends have been extrapolated, but often the prospective yields have been estimated by agronomists and other agricultural scientists who are well acquainted with a specific crop and area. The projections have been the work of competent scientists. It is clear that the yield projections have missed or will miss the mark by a substantial margin. It is also fairly clear that if development plans had been based upon these projections, serious shortcomings would have emerged in the prosecution of the plans or certain objectives would not have been fulfilled. For example, the number of farm families that could earn a given level of net income in agriculture would have been much greater than the number that can now earn that amount if the actual yields of crops had been at levels consistent with either of the two sets of yield projections for 1965. Prices received by farmers would have been substantially higher than at present. Had farmers been induced to make their plans in terms of anticipated prices substantially above realized prices, a large economic loss would have been realized.

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(1) The 1965 yield projections from the 1948 study were exceeded in 1954-56 in five out of seven cases and almost equalled in another.

B. *Projections for the Soviet Union*

In the Soviet Union the announced goals for agriculture represent goals both for production and for consumption, after adjustment for exports or imports and changes in stocks. Food consumption will also differ from production because some part of certain products is used for industrial purposes or part of the output is used as an intermediate product (feed or seed). For this brief discussion, the goals will be considered solely as production projections. The goals for the Sixth Five Year Plan (1956-60) and for the 1958-65 plan will be presented and compared with actual performance for the 1956-60 period and the first four years of the present plan.

Table 7 presents information for each of ten agricultural commodities - output in 1955 and 1960, the goal for 1960, the actual increase and the planned increase, and the actual increase as a percentage of the planned increase. Of the ten goals, only one was fulfilled. This was the goal for sugar beets and it was overfulfilled by a substantial margin; in fact, by such a wide margin relative to the fulfillment of the other goals that it might be said that there was an error in the execution of the plan (1). Of the other nine goals, the actual increase in output as a percentage of the planned increase in output ranged from 17% for cotton to 48% for wool. Instead of gross agricultural output increasing by 70%, the increase was only 32% or 46% of the planned goal.

The above comparison of goals and performance may be considered by some to be irrelevant since the Sixth Five Year Plan was abandoned in midstream. The Sixth Plan was replaced by the Seven Year Plan for the period 1959-65. While the Seven Year Plan is not yet completed — it is now in its fifth

(1) It should be noted that the procurements in 1960 were 5.2 million tons less than output. In 1955 the difference was only 0.4 million tons. The difference probably consists of both waste and feed. In addition, 5.9 million tons of sugar beets were produced for feed in 1960.

TABLE 7

Output of agricultural products, 1955 and 1960, goals for 1960, and increases in output and planned increase, Soviet Union

Product	Output 1955	Output 1960	Goal 1960	Increase actual	Increase plan	Actual increase as per-cent of planned increase
			(million metric tons)			
Meat . . .	6.3	8.7	12.7	2.4	6.4	38
Milk . . .	43.0	61.7	84.2	18.7	41.0	46
Grain . . .	107.0	134.4	180.0	27.4	73.0	38
Cotton . . .	3.9	4.3	6.2	0.4	2.3	17
Potatoes . .	71.8	84.4	126.5	12.6	54.7	23
Sugar beets .	31.0	57.7	47.2	26.7	16.2	165
Vegetables .	14.1	16.6	28.3	2.5	14.2	18
Wool . . .	0.26	0.36	0.47	0.10	0.21	48
Flax-fiber . .	0.38	0.42	0.51	0.04	0.13	31
			(billion units)			
Eggs . . .	18.5	27.4	46.7	8.9	26.7	33
			Index numbers (1955=100)			
Gross agr. output . .	100	132	170	32	70	46

Sources: « Pravda », February 26, 1956; « TSU, Narodnoe khoziaistvo SSSR v 1960 godu » (Moscow, 1961) pp. 362, 374-75 and 378.

year — the performance for the first four years can be compared with what would have been required if the goals were to be achieved. This is done in Table 8 by assuming that the output path between 1958 and 1965 was to have been linear (1).

For the four years ending in 1962 the annual increase in gross agricultural output was only 18% as large as that required to achieve the 1965 goal. Of the ten commodity groups, there were two with output declines. For the other eight, the ratio of the actual to the planned increase ranged from 21 to 93%, with only one of the ratios being in excess of 50%. For the commodity with the highest ratio of actual to planned increase — sunflowers — the planned increase for the period was only seven per cent.

The record of output performance of Soviet agriculture is somewhat better than that indicated by the estimated seven per cent increase in output between 1958 and 1962. In 1958 climatic conditions were very favorable and agricultural output was probably five per cent greater than it would have been under average climatic conditions. However, in the published discussions of the Seven Year Plan I have seen no evidence that the effect of the favorable conditions in 1958 was taken into account in establishing the 1965 goals. I suspect that the modest increase in planned output for sunflowers and other oil-bearing seeds reflected the high 1958 yields, but I have seen no place where this was recognized.

A brief comment concerning the decline in sugar beet output between 1958 and 1962 may be in order. There is little doubt that the output of sugar beets for refining could have been larger in 1962 than it was. The sugar beet goal for 1965 appears to be an instance where the goal was set too high in terms

(1) According to data presented in a speech by N.S. Krushchev (« Pravda », March 5, 1962) it appears that the annual goals for agricultural products for 1958-65 are approximately linear interpolations of the 1958 output and the 1965 goal. For two examples given — grain and milk — almost exact linear interpolation was involved; for the third example — meat — output was supposed to have increased at a greater absolute amount in earlier than in later years.

TABLE 8

Output of agricultural products, 1958 *and* 1962, *goals for* 1965, *and average annual increase* 1958-62 *and planned increase,* 1958-65

	Output 1958	Output 1962	Goal 1965	Average increase actual	Annual plan	Actual increase as percent of planned increase
			(million metric tons)			
Meat	7.7	9.4	16.0	0.42	0.97	43
Milk	58.7	64.2	100.0 (1)	1.37	5.34	26
Grain	141.2	147.5	164.0 (2)	1.50	3.26	46
Cotton	4.34	4.50	5.7 (3)	0.04	0.19	21
Potatoes	86.5	68.6 (4)	147	−4.48	8.64	— (8)
Sugar beets	54.4	47.2 (5)	76 (6)	−1.80	3.09	— (8)
Wool	0.32	0.37	0.55	0.01	0.03	33
Flax fiber	0.44	—	0.58	—	—	—
Sunflowers	4.63	4.8	4.95 (9)	0.04	0.05	93
			(billion units)			
Eggs	23.0	30.2	61	1.80	5.42	33
			Index numbers (1958 = 100)			
Gross agr. output	100	107 (7)	170	1.75	10.00	18

Sources: « TSU, Sel'skoe khoziaistvo SSSR » (Moscow, 1960), pp. 27, 31, 202-03, 332-33 and « Pravda », January 26, 1963.

(1) Minimum goal; maximum goal is 105 million tons.
(2) Minimum goal; maximum goal is 180 million tons.
(3) Minimum goal; mamixum goal is 6.1 million tons. Goal and output in terms of seed cotton.
(4) Output in both 1960 and 1961 was 84 million tons.
(5) 22.9 million tons of sugar beets were used for feed; these amounts probably could have been processed for sugar.
(6) Minimum goal; maximum goal is 84 million tons.
(7) Estimated from official data on output by commodity groups; weighted by 1958 prices paid to collective farmers.
(8) Output declined between 1958 and 1962.
(9) The plan specified a goal for oil-bearing seeds; I have assumed the goal for sunflowers was 90% of the goal for all oil-bearing seeds on the basis of 1958 output of sunflowers and other oil-bearing seeds.

of consumption requirements or demand when account is taken of the quantities of other foods available. This probably explains why there was a substantial diversion of resources from producing sugar beets for refining to sugar beets for feed. The large imports of sugar from Cuba have probably had a significant influence on the decision to reduce the refining of sugar beets.

In 1961 the Soviet Union imported 3.34 million tons of sugar (refined basis). Total granulated sugar production in 1962 was 7.8 million tons, of which 6.0 million tons was from sugar beets. Total sugar refining in 1962 was 7% below 1961. The decline in refined sugar from 1961 to 1962 is consistent with the view that the 1965 sugar goal was set at too high a level relative to other farm products.

One of the factors responsible for the modest increase in Soviet agricultural output since 1958 has been the failure of industry to deliver the inputs for agriculture that were implied in the plan. In March, 1962 Krushchev remarked as follows (1):

« The seven-year plan called for increasing the output of mineral fertilizers from 12,000,000 to 35,000,000 tons, an increase of 23,000,000 tons. Three years have passed, and the production of fertilizers has increased by only 2,900,000 tons. In the first three years of the seven-year period the plan for new capacity fulfilled only 44%. The same thing is happening with the organization of herbicide production. Two years ago the Central Committee and the government adopted a decision on this question. Time is passing but there are no herbicides. »

The 1962 fertilizer output was 17.3 million tons or 13% greater than in 1961. Fertilizer output in 1958 was 12.4 million tons, instead of the 12 million tons implied by the above quotation. Thus in four years the increase in output was only 4.9 million tons.

(1) « Pravda », March 6, 1962.

In his March, 1962 speech KRUSHCHEV also noted that the deliveries of certain farm machines declined substantially between 1957 and 1960. However, there appears to have been a substantial increase in farm machinery production since 1960 - perhaps by 50%.

C. *General Comment*

It is fairly obvious from the above brief discussion of projections, goals, and performance that much remains to be achieved before specific projections of output and consumption are to provide an adequate base for development plans. It can be argued, of course, that either inadequate analytical and statistical methods were used or that the goals were politically motivated. Such an argument could be correct, but it implies that more adequate methods are available and could be applied in specific cases and that the political elements can be eliminated.

IV. SOME ANALYTICAL AND STATISTICAL PROBLEMS

For short periods of time, say three to five years, it appears that fairly accurate projections of changes in the demand for agricultural products are possible. However, even this statement must be qualified to exclude such effects as the Korean War or a significant change in the policies of other importers or exporters where international trade is involved.

Projections of changes in output or of the effect of specific policy measures upon output are subject to much greater uncertainty. Even if we exclude the problems that may arise because of climatic variations, other determinants of output do not appear as yet to be subject to reasonably adequate projection. This is true both of aggregate agricultural output and of the output of individual farm products.

The factors that affect changes in output are often extremely complex and not too well understood. In fact, some are of a sort that are virtually unknowable. How do we know when a new variety of seed is to be available? How do we know how rapidly it will be adopted by farmers? How do we know how quickly it will be improved by more adequate adaptation to local conditions? In the United States we have learned a substantial amount about the answers to the last two questions from studies that have been made of the adoption of hybrid corn ([1]). While we knew that a new hybrid for another crop would probably be adopted much more quickly than the 13 years it took before corn hybrids changed from 1 to 80% of the seed used, we were not able to predict that the same change would occur in grain sorghums in about 3 years. Nor did we predict that the yield advantage would be as great as it was. Our original expectation was that the yield differential between hybrid and ordinary grain sorghums would be about 25%; the actual yield differential appears to exceed 50% and may be as much as 75%.

The use of trends to project output appears to be of little value. Starting with the second decade of this century, the decade increases in U.S. farm output have been as follows (in percent):

1910-19	8
1919-29	13
1929-39	10
1939-49	28
1949-59	18

([1]) ZVI GRILICHES, *Hybrid Corn: An Exploration in the Economics of Technological Change*, « Econometrica », 25 (4), October 1957.

The thirties was a period of adversity on agriculture — drought, low prices, low incomes — yet output increased more than during the 1910-19 period and almost as much as during the twenties. Then during the forties output increased almost as much as during the two previous decades. While the growth of output was smaller during the fifties than during the forties, the growth was greater than in any of the first three decades included in the tabulation and despite certain efforts to restrict production.

The same kind of seeming discontinuity in output growth can be illustrated by changes in output in the Soviet Union. Between 1953 and 1958 the official index of gross agricultural output increased by 49% (1). This was an annual compound rate of growth slightly larger than 8%. Between 1958 and 1962 gross output increased by approximately 7% or at a rate of less than 2% annually. While most Western observers did not expect the growth rate achieved between 1953 and 1958 to be maintained, I do not believe that there was any one who predicted a growth rate as low as the actual one. Mr. ARCADIUS KAHAN and I predicted that output might increase by about 24% between 1958 and 1965 or at an annual rate of about 3%. Our projection should be compared with the increase of 70% indicated by the Seven Year Plan.

Ex post we can say a great deal, both for the United States and the Soviet Union, as to why there have been such changes in the rate of growth. But in making projections we are still not in a position to do very much in the way of predicting changes in methods of production or the effects of changes in incentives. For example, not all of the differences in views concerning the effects of the output price level on production in the Common Market or in the United States are due to self interest or political views; a large part of the differences exist

(1) I do not believe that the official output data for 1958 and 1953 are strictly comparable, but the relative overestimation of output in 1958 is not so great as to negate the point made in the text.

because we do not yet have an adequate analytical or statistical framework for estimating aggregate supply response of agricultural products.

In many of the underdeveloped areas of the world, data are either lacking or of unknown reliability. If it has so far proved difficult to make reasonably accurate projections for the United States or the Soviet Union, one can hardly be sanguine about the possibilities for Africa, Asia, or South America.

I do not want my remarks to be interpreted as implying that all efforts at projections are futile nor that projections even though subject to substantial error are without value. It is certainly important for a nation that it be aware that if present policies continue then it is quite likely that farm output will grow no more rapidly than population, for example. Such a projection should not be interpreted that food output will not grow more rapidly than population, but that to assume otherwise may well result in an undesirable consequence in the future and that alternative policies should be considered and evaluated.

But I do not believe that projections that can now be made possess the necessary degree of accuracy to make it feasible to rely upon detailed planning procedures which largely supplant the operation of the market. There are many limitations in the way the market functions, but there is abundant evidence that there are also many limitations involved in the execution of detailed and centralized plans.

There is a strong and convincing case for many types of governmental action that will make economic growth more rapid and less costly in terms of current consumption. Included in such actions are certain obvious candidates — primary and secondary education for the rural population, research and adult education, improved market information, sanitation and health measures. These are measures that improve the quality of the human agent and provide the rural population with the means for rational decisions. In particular situations, land reforms, governmental measures to improve and expand credit facilities, subsidization of specific production practices as a means of speeding their adoption (including the creation of the capacity to produce such items as fertilizer), special measures to bring new lands into cultivation through irrigation, drainage, clearing or through provision of roads and other facilities that would make it possible to settle new areas or price supports at moderate levels as a means of encouraging the expansion of commercial production merit appraisal and consideration.

The underdeveloped areas are primarily concerned with achieving an increase in the rate of growth of agricultural output. This must be accomplished in a setting in which the resources under the control of the government are relatively limited. The funds that can be invested and the trained personnel available for carrying out a plan or policy are clearly very limited. In many countries there are not enough trained economists and other specialists to develop the analyses and estimates required for the formulation of a detailed development plan or policy. In some cases, foreign specialists can assist in such formulations, but the value of foreign experts can be easily overestimated.

One of the most important contributions that econometric analysis can make to the underdeveloped economies might well be a series of studies that will aid in the decisions involved in the allocation of trained personnel. Will such personnel have a higher marginal product in establishing a research and extension program or in organizing an irrigation project? Will the marginal product be greater in developing a series of investment priorities in the agricultural sector or in analysis of the tenure system and other factors affecting incentives? These, and similar questions are extremely difficult to answer and are generally not the type of problems tackled by econometric methods. But such questions may be more important than some of the questions that we are ordinarily interested in.

In the United States and Western Europe the major policy objective concerns the level of farm incomes. The major analytical questions are not the same as the ones discussed above. The questions relate to the social costs of alternative policies and the effects of alternative policies upon the level and distribution of farm income and the size of the farm population. There does not now exist, for example, a convincing analysis of the effects of reducing the wheat price paid to German farmers by 150 Dm per metric ton upon the average level of return to labor and capital engaged in agriculture. Nor does there exist an adequate analysis of the effect of the U.S. farm programs upon the level and distribution of farm incomes or of the effects of substantially reducing that level of expenditure.

Basically, our difficulty is that we have not as yet been able to satisfactorily estimate the aggregate supply function for agriculture. Given the many variables that affect the aggregate supply function, when input prices must be allowed to change as output prices change, when factor supply functions are shifting over time and adjustments to changing factor and output prices do not occur instantaneously, it is perhaps understandable that we have thus far made so little progress.

DISCUSSION

MAHALANOBIS

I am practically in agreement with all the points made in this paper. I should just stress two or three points which seem to me to be of particular importance. Firstly, industrialisation as an essential condition for improvement of agricultural production. There has been a lot of unnecessary controversy in a country like India by raising the question which must you emphasise more, industry or agriculture? Both of course; there is no conflict between industry and agriculture. It is a simple but an extremely important point.

Secondly, I was very much interested to see, on page 19, the estimate of rate of growth of 3.5 per cent to 4 per cent in agriculture; such a rate of growth is absolutely necessary for economic development. In India it is necessary to have a rate of growth of the order of 7 per cent per year for the economy as a whole. With population increasing more than two per cent per year, the per capita income would increase at the rate of 5 per cent per year and would double roughly in 15 years which would give some cause for hope. To achieve this it is necessary to have a rate of growth of the order of 4 per cent per year for agriculture.

I should also entirely agree that demand estimates can be extremely useful; we have found this in India. In calculating income statistics, we have used a special method of arranging the sample households in ascending order of level of living, as measured, roughly, by per capita consumption expenditure, and dividing the sample (or the estimated population) into a suitable number (say, 100, 50, 25, 20, 10 etc.) of groups, called « fractile groups » each consisting of the same number of percentiles (1%, 2%, 4%, 5%, 10% etc.) of the sample, and then finding elasticities in terms of the fractile groups. These elasticities (or their ratio) seem to be fairly steady; by putting in current prices one can then immediately get income-elasticities in money term at any point of the range.

Some interesting points have come out in India about the pattern of change of consumption (in physical or money terms) of certain commodities with increasing levels of living as measured by the per capita consumption expenditure. For certain commodities this pattern is quite different in urban and rural areas; for example, for foodgrains. The interesting point is that, over a period of 10 years, the pattern of consumption of foodgrains in urban areas and the pattern in rural areas have been entirely different, but both have remained practically steady. With the increase of income (in the sense of national income or of households) in a country like India which is very poor, or with an improvement in the supply position of foodgrains, the per capita average consumption of foodgrains of the whole population may go up or some times, with increasing prices, may go down, but the pattern remains the same. Of course we are interested in this, particularly from the point of view of inequalities. The National Sample Survey of India, which covers the whole of India, has at least two inter-penetrating sub-samples giving two independent estimates; and also, of course, a combined estimate by pooling the sub-samples. Using a graphical representation (in the form of distribution of per capita consumption, or concentration curves or in other ways) the difference between the curves based on the two sub-samples gives a non-parametric and completely invariant error area with which the analysis can be carried very far; I discussed it in another paper. From our own experience I fully agree with the points made by Prof. JOHNSON in his paper.

As regards the supply position, I think in an underdeveloped country like India, the official figures are often unreliable for foodgrains. There is no concept of checks and cross-checks. It is due to treating statistics as legal evidence, that is, to consider that its sanction depends on the level of authority which gives administrative approval. The Food and Agriculture Ministry in India gives some estimates of foodgrains; as this Ministry is in administrative charge of matters relating to foodgrains, its estimate is the only one which must be accepted (in the sense of a law court accepting legal evidence). Even if another Ministry or Department of Government independently make some other estimate (say, from consumption side) then that estimate is not « official » in the legal sense and cannot be used. In fact, in India, it has been urged that only one agency should collect each type of statistics because multiplicity of agencies might lead to differing estimates which would be confusing; in other words, the very possibility of having checks and cross-checks must be eliminated. It is a somewhat paradoxical situation. India has some kind of reputation outside India for its statistics; but Indian statistics remains weak because statistics is treated as a matter of legal evidence and not of scientific validity. The official figures of foodgrains in India (based on so-called complete enumeration of all agricultural holdings), in my personal judgement, may be underestimates by some thing of the order of 20 or 25 per cent. The production of wheat and rice and other foodgrains can be estimated through samples on the basis of the area sown together with the yield per hectare ascertained by crop-cutting experiments. The consumption of foodgrains can also be ascertained through a sample of all households in the country. A direct comparison is then possible between such estimates of consumption with the independent estimates of production after allowing for seeds, inventory, loss in storage etc. As there would be at least two sub-samples and two independent estimates of both consumption and production, such comparisons can be carried out in an entirely scientific way. In the case of a cash crop like jute, during the war, the production estimates given in October of one year could be verified about 15 months later from the utilization estimates based on figures for shipping of both raw jute and manufacture and changes in stock in the hands of jute mills; the production and utilization figures were in agreement within a margin of about 2 per cent. Although such an objective method can be used, the so-called « official » estimates of foodgrains in India are based on the perfectionist theory of figures being obtained for every field in the village; because this is a so-called complete enumeration, and because such a system has been in use for centuries, therefore it has high legal status like custom immemorial; and also no independent check should be permitted, because it would be confusing if the results of such a check differed from the official estimate. So the point stated by Mr. JOHNSON is well taken and extremely important. It also raises the question whether a lot of sophisticated acrobatics on econometrics on the basis of such unreliable statistics in the underdeveloped countries would not be a complete waste of scarce resources.

I entirely agree that without progress of agricultural output no rapid economic development is possible. I do not clearly understand the last paragraph at the end of page 37 « I do not believe that projections that can now be made possess the necessary degree of accuracy to make it feasible to rely upon detailed planning procedures which largely supplant the operation of the market ». I understand what this sentence means, but on the other hand, I take it that the next paragraph is making out a case for State intervention. I find a little gap here. Can you always leave it to market operation as such in an underdeveloped country? I should strongly differ. In my own country in 1943, during the war, because of that particular doctrine (I may mention that the Economic Adviser of Government at that time was Dr. GREGORY, the author of a standard book on the Gold Standard) there was no attempt at any control or rationing of foodgrains with the result that a famine broke out in Bengal. An official commission appointed by the British Government found that one and a quarter million people had died directly from famine, which is about twice the total casualties of the U.S.A. and U.K. taken together during the Second World War. This happened. Therefore, I do believe that government intervention is essential but should be a minimum.

[16] *Johnson* - pag. 44

With the points made in pp. 37-38, I am in complete agreement We are rapidly promoting the use of fertilizers in India; we are also starting the manufacture of machinery to set up new fertilizer plants. But there is one point which has not been brought out, namely, the need of price support, or more precisely, having both floor and ceiling prices because consumers must also be protected. There should be buffer stocks and open market operations to the largest extent possible. In India, Government could hold 5 or 6 million tons of foodgrains in suitably located warehouses. There would be pre-determined floor prices which may vary for different crops in different areas; when the price touches the pre-determined low point in any area, Government would start purchasing immediately with a guarantee to continue without limit; this would give necessary price support to growers. On the other hand, once the ceiling price is touched, Government would start selling and this would protect the consumers. Such a system, I think is necessary in countries where there is a scarcity of food but not in countries like Thailand or Burma where there is a surplus. I think it would be of great help if the advanced countries would try to persuade an underdeveloped country to adopt the above policy (using if necessary United States PL 480 arrangements) to build up buffer stocks and have open-market operations with not too large a gap between the floor and ceiling prices so that speculators would not come in. This should be adequate; other methods of physical control may have to be used if the scarcity becomes very acute. I am amplifying the remedies but not really differing from Mr. JOHNSON.

I could not agree more with the statement on page 38, that there are many questions which are not now within the range of econometricians, or may be they can never be — I am leaving that question open; I am simply agreeing that they are now beyond the techniques which have been so far developed and which may be entirely suited to the needs of the advanced countries. There is a great danger of highly trained econometricians in my country going on solving problems which may be entirely valid in respect of advanced countries but which have no relevance to the problems of an underdeveloped country. I see a great danger here. On the other hand, there are very important questions, some of which have been raised by Mr. JOHNSON, about how the econometricians in advanced countries can help the underdeveloped countries. This is a question of vital importance, not only for the underdeveloped countries, but also in the enlightened self-interest of the advanced countries.

KOOPMANS

A brief question — in these elasticity estimates, does food include the tin can and the tip to the waiter, or not?

JOHNSON

What I talked about was food at the farm-gate level.

KOOPMANS

The other question is about the need for these projections — so to say the marginal productivity of improvements in the accuracy of the projection. What decisions do depend very much on having accurate projections and what harm is done if projections are inaccurate? If no great harm is done it is a nice purpose just the same to improve accuracy, but it wouldn't rank with as high a priority in econometric work.

JOHNSON

I think this is a very pertinent and relevant question and I would try to answer it in terms of the difference in plans in the planned and the essentially free-market countries. I think that incorrect projections, if they affect policy decisions, can sometimes have serious consequences. I would like to give two examples, one drawn from the U.S. and the other from the U.S.S.R. In the early part of the last decade, about 1950, a series of projections about future demand/supply in agriculture were made for the U.S. The projections implied that the growth of demand over the next decade was going to be very rapid and that there seemed to be little possibility of rapidly expanding production beyond what had been achieved during World War II. It is impossible to say the exact connection between that series of projections and what was actually done in the policy field; it at least was used as an excuse in the discussion for increasing the support prices in the early 1950s; this action had very serious economic consequences, later, in inducing too many resources to remain in agriculture. Obviously if nobody had paid any attention to these projections, they would have done no harm, but in this case attention was given to them and we arrived at the wrong decision.

In the U.S.S.R. the Seven Year Plan for 1958-65 involved projections for agriculture. These projections or plans were made with great enthusiasm, but I assume were believed to be realistic. A large increase in output was to be achieved by 1965, largely through capturing what Soviet officials call « reserves » which didn't require additional resources; thus it wasn't really necessary to push the fertilizer plan or other plans to increase the resources used in agriculture. The errors in the projections have had a serious impact. Had the official projections for the U.S.S.R. been similar to those made by Kaban and me which indicated that output would increase by less than 25 percent instead of 70 percent in 7 years, I suspect that several very different policy decisions would have been made during the past few years. Some of the difficulties that have resulted from a reduction of the grain crop by perhaps a quarter in 1963 compared to 1962 would not have occurred. The U.S.S.R. would not now be using perhaps a billion dollars in foreign exchange to purchase grain during the coming year (1963-64); they would have taken policy measures that would have prevented these expenditures.

MALINVAUD

Professor JOHNSON has shown us that long-term forecasts in agriculture were often very poor. I should submit that this is not surprising. The future is difficult to predict anyway; and quantitative economics is not an old science. Economist, who only recently started to work with real data, have still much to learn in order to predict better.

Even considering the present achievements, I would not accept the conclusion that we should no longer try to make long-term projections. Whoever has to decide for the future, must have some idea of the future. Thus, the real alternative is between individual long-term projections and what I would call centralized long-term projections. Professor JOHNSON has not convinced me that we should only rely on individual projections.

One of his claims is that individual errors are offsetting. But, in the first place, I doubt whether they really are. Great mistakes have been made during the past 15 years in various industries. My feeling is that, in most cases, the centralized projections were less wrong than the average individual projection. The centralized projection had to fight against the common belief that was generally much too extreme. Such was the case for fuel during the European coal shortage, and again shortly after at the time of the Suez crisis. According to me, econometric studies are usually bringing into the discussion about the future some rational elements which are not taken into account by individuals who have no time to go into a serious analysis.

In the second place, even if the errors were offsetting, we should not necessarily be satisfied with the situation. Offsetting errors do not imply good decisions. Each individual decides on the basis of his own mistakes. Decisions will then be inconsistent with one another to the extent that they are based on different assumptions about the future. To convince me, one ought to prove that such inconsistencies have no detrimental consequences.

In line with Professor MAHALANOBIS, I should add that I do not like much the expression « to supplant the operation of the market » because it implies more than it really says. Most people would agree that we have not to supplant the operation of the market if the market operates well. The only question is to know whether it does operate for the long-term allocation of resources. Are long-term decisions in agriculture enlightened by the operation of the market? So far as I know, they are not. Nowhere in the market process will an individual banker who considers making a loan to a farmer, find when the hybrid corn will reach the 80% level of diffusion, or what the price of corn will be in five years from now.

JOHNSON

There is no question that some of the remarks in the last three or four pages are quite cryptic and perhaps warranted more detailed exposition. My excuse is that I thought the paper was already too long. I would like to explain and — in some degree — defend the position that I took with respect to the usefulness of agricultural projections and that such projections do not now « possess the necessary degree of accuracy to make it feasible to rely upon detailed planning procedures which largely supplant the operation of the market ». Both Prof. MAHALANOBIS and Prof. MALINVAUD were somewhat concerned about that statement.

What I meant by « detailed planning procedures » were quantity allocations, of inputs or outputs, or price determinations by governments that effectively determine the prices that affect producer and consumer decisions. Obviously, there are a variety of interventions — planning — where I feel that we know enough to rely on governmental actions; these are listed in the paragraph following the sentence that has elicited these reactions. I there refer to education, research, land reform, building roads. These are actions that supplant the market; when a government builds a road, this supplants the private market for the road.

With respect to Prof. MALINVAUD's other points, their basis may be in the brevity of my presentation but I am not sure. On page 37 I state specifically, « I do not want my remarks to be interpreted as implying that all efforts at projections are futile nor that projections even though subject to substantial error are without value ». This is rather different than what it has been implied that I said. In the paragraph from which the above sentence was taken, I noted that if projections indicated that food output was likely to grow less rapidly than popnlation, governments should clearly act in response to such a projection if it is believed to have a sound basis. In the United States, all projections indicate that if present policies are continued, output is going to grow more rapidly than consumption. Such a forecast has significant policy implications.

I did not say that « individual errors are offsetting ». What I did say (on page 22) was the following: « For one thing, errors made by private individuals may be offsetting. For another thing, errors made by private individuals may bring into play forces to correct the error, such as a decline or increase in market price, while a government price policy, subject to rather more slowly functioning political processes, may compound the consequences of projection errors. »

I am particularly concerned that Prof. MALINVAUD has interpreted me as saying that simply because private forecast errors may be offsetting, the resulting resource allocation is an efficient one. I did not say this; in fact, I once wrote a book (*Forward Prices for Agriculture*) about the problem and concluded that a type of governmental price forecasting for a production period could lead to an improvement in resource efficiency.

But the point I have tried to make was that where errors are made by private individuals, and this is particularly true in agriculture, the market does bring into play forces that correct the errors, whereas, in most cases where government policy is involved the impact may well compound the effects of the previous error.

German agricultural policy today is an example of such a response. Farm prices have been too high in the past and are too high now. The high prices are impeding certain adjustments that could well occur within the German economy in transferring labour out of agriculture. However, since income growth in agriculture was not quite as rapid as in the rest of the economy, the solution adopted two or three years ago was to increase farm prices even further. The basic problem is that there are too many people engaged in German agriculture. So the error that was made by not trying to make the adjustment through helping to transfer resources out of agriculture has resulted in a compounding of the difficulties.

MALINVAUD

As I see it, that example does not show the government should not interfere. It illustrates a rather different point, namely that it is sometimes a difficult affair to determine first what a wise policy should be, then to enforce it consistently, and also to know how to revise it when necessary.

ALLAIS

I have only two very brief points. The first one is that I would like to stress the interest of the work done by Professor JOHNSON in recent years about agriculture in Soviet Russia. This work is very useful and I had in the past the occasion to use it and I think that is a type of study which should be multiplied. My second point is, I think, quite important, in view of the discussion we must have for the final statement. I must say that I agree completely with what Prof. JOHNSON said on page 3 of his paper, namely, that it appeared to him that it was necessary to paraphrase the introduction to this colloquy and I feel great sympathy with what he says. To quote him: « The application of economic analysis and recent experiences of various economic systems have by now given sufficient proof that modern or underdeveloped agricultures cannot be subjected to agricultural development plans that impede necessary adjustments, result in exploitation of the farm population, fail to recognize the potential high returns to certain investments in agriculture, or restrict the potential gains from international specialization in production, but development plans must be consistent and efficient in the use of scarce resources and not result in undesirable economic situations. » I would say this judgment is valid not only for agriculture but also for industry and for the whole economy, and I hope that we could stress this in our final statement.

HAAVELMO

I would like to ask Prof. JOHNSON if the need for the kind of research that he has suggested, might not be somewhat different in areas with a low density of population and in areas with a high density of population? It seems to me that the kind of production policy needed may be different in the two cases.

WOLD

What I want to say has partly been said by Prof. ALLAIS. First of all I wish to compliment Prof. JOHNSON for his superb paper, and to emphasize that his general conclusions make important material for our final statement.

The eminent qualities of Prof. JOHNSON's paper invite to general comments on the position of present day econometrics. Econometrics is still a young science. Speaking broadly, there has been a gradual evolution and expansion from micro to macro approaches. Demand analysis and the assessment of cost functions and production functions are key areas of theoretical and applied work where econometric methods have been well established for a long time and where they are in every day use both at micro and macro level. Macro problems, again speaking broadly, are on the whole more complex and difficult, and the econometric methods are here far from having the firm grasp on the problems as at the micro level. The theme of our Study Week is an inspiring challenge to report on the present status of econometrics in the treatment of economic growth and business cycles, the two major areas of macro analysis. It is safe to say that as the avenues of quantitative analysis and econometric techniques have broadened in these areas, it has gradually become more and more clear that the two groups of problems involve a highly complicated network of interrelated phenomena, the conclusion being that the problems are not amenable to simple and easy solutions. The spearheads of research and applied work have moved forward in significant steps, marked by successful attacks on a number of partial problems, but I think everybody will agree that the successes have as yet only been partial. More specifically, no integrated model of business cycles or economic growth has as yet been reported which has been successful in the qualified sense of passing the purgatory of a strict predictive test (see the third section in my report « Toward a verdict on macroeconomic simultaneous equations » to the Study Week). Well to note, the emphasis of this statement is not on the shortcomings of existing models — personally I am convinced that the road is well paved for continued progress, and the goal of reliable predictions will be reached in due course — on the contrary, the emphasis is on the partial results thus far established. It seems to me that econometrics by now has reached an intermediate stage between micro and macro. Micro analysis is well consolidated with regard to problems, methods and results; Professor Johnson's brilliant report is ample evidence that econometrics is now mastering macro analysis in the sense of economy-wide approaches, and it is appropriate to see this as an intermediate level of macro analysis since he is dealing with the agricultural sector, thus making a partial and not a total analysis of the economic system. In the programme of the Study Week there are several other important contributions at an intermediate level between micro and macro analysis. Specific reference is made to reports that focus on methods and techniques of general scope; a typical case in point is the device of shadow prices dealt with by Professor DORFMAN.

KOOPMANS

I have a question to Prof. JOHNSON in response to his comment on investment in research. Research and development in industry is often exportable to underdeveloped countries. My question to him is whether this is also true in agriculture, or whether because of the specificity of climate, soil and varieties, agricultural research has to be essentially done over for each agricultural area.

LEONTIEF

Professor JOHNSON's penetrating observations on the role which econometric models might play in advancing the growth of underdeveloped areas and possibly even of developed countries naturally lead to the question of choice between special and general purpose models. Throughout our present discussion much stress was put on the necessity of building special models for special purposes on the one hand, and on the other hand it was emphasized that the same model can serve several different purposes.

Most of the difficult economic problems are those which involve discovery and tracing through unsuspected secondary relationships between the different parts of the economic system. A policy maker, as a rule, is able to assess correctly — even without any help from the econometrician — the obvious direct effect of measures which he is about to recommend. Quite often he does neglect or disregard, however, the indirect effects which might be unimportant from the point of view of his immediate objectives, but which might be in conflict with different but possibly equally important objectives pursued simultaneously by other policy makers.

The principal advantage of general planning and overall projections, as compared with haphazard promulgation of special purpose policy measures, is that it permits us to avoid such conflicts.

Special purpose models which permit us to estimate the principal, or should I say, most desired effects of one particular policy measure, but slurs over its secondary repercussions is too blunt an instrument. Consistent policy formulation requires general purpose, which is essentially general equilibrium models.

JOHNSON

I think there are only three of the statements that necessitate a response from me. The first was Prof. HAAVELMO's question, asking whether the research needed would be somewhat different for areas of low density than high density populations. I would argue the difference would be fairly small and the reason is the following: in the low density areas, which include large parts of South America and Africa, an expansion of output which is based upon using additional land which can be brought into cultivation without any change in technology will not lead to any significant increase in per capita levels of income. In my paper I described what happened in the United States from 1820 to 1860, when the U.S. clearly had low-density of population; agricultural output grew rapidly but as near as we can tell output per worker changed very little in agriculture; and also, although this does not necessarily follow, this was a period when per capita income in the United States did not increase to any significant degree. But to repeat, I would say that in the low population density countries they also need substantial increases in productivity per man and per acre and that many of the factors that are crucial to prevent a decline per capita income in the high density countries also apply there.

Mr. KOOPMANS' question is a very pertinent one and that is, whether research in agriculture is exportable; can the research done in the United States or Western Europe be exported to India? I would say the fundamental scientific research such as the theory that has led to hybridization is clearly exportable, but I fear we cannot go very far beyond that. It is necessary to adapt varieties, different types of plants and even the methods of fertilization to local conditions. There is also one other factor that is important here and this is that in agriculture there is a continual fight between man and nature. I would say that upwards of 3/4 of all the research on the small grain in the United States done today is purely an action required by man's fight against nature. The resistance of various plants to insects, disease, and viruses disappears over time and there is a continual fight just to maintain yields; in wheat, for example, in the United States, almost all research over the last thirty years has been to fight against nature. A wheat variety that is very popular in the United States and very high yielding might give a zero yield after two or three years in India because it became subject to an insect or to a disease.

I am not quite sure how I should respond to Mr. LEONTIEF; perhaps no response is required other than that I agree with his view that a general purpose or general equilibrium model that would do all that special purpose models do — and much more besides — would constitute a major contribution to both analysis and policy. But after saying this, I must note that I do not believe that as a profession we have progressed so far in the empirical application of general purpose models to permit us to abandon the generation of special purpose models.

Obviously special purpose models must be used with care if important policy conflicts are to be avoided. However, two decades of observation of agricultural policy in the industrial countries does not convince me that the present unsatisfactory state of agricultural policy has been due to a failure to understand the « indirect effects » of the measures adopted.

SELECTION AND IMPLEMENTATION
THE ECONOMETRICS OF THE FUTURE

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What I am going to present to you today is in all humility a frontal attack on a ghost that has been haunting all of us for the last generations, whether we want to be classified as belonging to the West or to the East or to the uncommitted countries, which, with a few exceptions, are the countries in Afro-Asia and partly in Latin-America that are now striving towards rapid economic and social development.

I can do nothing better than to begin by quoting the introductory part of the program of this Study Week. And, incidentally, this introduction is a significant indication of the profound understanding of the basic problem of our times which the organizers of this Study Week, with the blessing of His Holiness the Pope, have had. The introduction begins with these words, and I quote:

« Modern economies are extremely complex and both theory and practice show that the free play of individual choice does not guarantee, as used to be thought, favourable results for the community.

Once this is admitted it is obviously necessary to provide suitable informative and control instrument and fix the targets which the economy is aiming at ».

I think it is fair to say that the free market system has two advantages: (1) its simplicity and (2) its effort-releasing effect. But it has one fundamental shortcoming: it does not assure the realization of *specific* preferences, such as a high rate of economic growth, a distribution of income and wealth based on social justice, aid to special social groups, economic development of lagging regions within the country, development of special agricultural and industrial sectors (for defense, health or humanitarian reasons) etc. The purpose of wise planning is to realize many such special goals, while retaining as many as possible of the advantages of the competitive system.

We wish to search for some better economic system to replace the time-honored system of the free market economy. But in that search, we encounter a ghost that has been haunting all of us for the last generations. It has been the same ghost we have encountered regardless of the direction we have chosen in our search for a better economic system.

This ghost is human nature itself. Some people are alert, full of initiative and driving force, full of the active and unselfish desire to apply all their abilities to the economic and social betterment of their country and to that of mankind as a whole. But, alas, the percentage of people possessing these virtues is small, very small indeed. Many people are, more or less, dull and selfish and can be induced to make a personal effort only if thereby they can obtain some tangible advantage for themselves or to the people close to them. In this connection, the economic advantage will often stand in the foreground.

Therefore, the historical challenge, facing us as economists and social engineers, is to help the politicians work out an economic system built upon a set of incentives, under the impact of which the economic activity will be satisfactory from the viewpoint of the economy as a whole, even if the behaviour of many individuals is essentially selfish. We must find a means of circumventing the human obstacle to human progress.

It is this means about which I am going to speak today.

To size up the nature of the problem, let us review briefly the directions that have been explored in the search for a better economic system. Roughly speaking they may be classified into three groups.

The first direction, one characterized by a very mild deviation from the traditional market type of economy, consists of admitting only monetary and fiscal instruments in the attempt at steering the economy. The human behaviour patterns at the various levels of society are such that, in this mildest form of attempt to steer the economy in a desirable direction, one faces a fundamental choice between inflation accompanied by fairly full employment or a reasonably stable price level accompanied by less than full employment of labour and other resources. A precise description of the situation would, of course, necessitate specifications of a number of details, but the choice I have mentioned indicates the essence of the matter. This choice is strikingly illustrated in the famous Samuelson-Solow menu. This menu consists, as you know, of a curve applicable to the United States' economy and showing how rapid an increase in the price level we must be willing to accept in order to reduce the unemployment percentage to a given level. An even more important fact is that monetary and fiscal instruments alone are not sufficient to assure the fulfillment of the highly *specialized* preferences we may have regarding the results to be obtained from the community's economic activity. The mild form of steering about which I am now speaking might perhaps be described by saying that it is a timid attempt to introduce a small amount of enlightenment into that which I have called, on several previous occasions, the unenlightened financialism.

The second direction in the search for an improved economic system deviates a little more from the traditional market economy. It consists of admitting state intervention of various sorts, aiming at influencing *directly* the quantities of goods and services produced or consumed. Among the attempts in this direction fall numerous sorts of quantity regulations — in particular, state control of investment in physical capital. In some cases, this has been combined with nationalization of existing big enterprises and/or with the establishment of new state-owned or state-controlled enterprises which are to operate alongside the still remaining private enterprises. A common characteristic of all these arrangements has been their operation under a monetary and financial machinery which, in all essentials, was to remain of the traditional type, meaning that we are still confronted with the Samuelson-Solow menu and facing tremendous administrative problems, including problems of loyalty and morale. These, I think, are in a nutshell the characteristics of the mixed economies seen emerging in many countries today.

The ghost has performed in his typical manner in all these pursuits. Quantitative regulations of the prescriptive type have a tendency to kill initiative and make the activity inflexible, inefficient and stationary. And state administration, because it takes away both the stick and the carrot which function under a hard competitive system, has a tendency to eliminate a large part of the driving force for personal effort.

Finally, the third direction, in search of a better economic system, is represented by the more spectacular deviation from the traditional market economy which is found in the centrally steered economies of the East.

This more radical departure from the traditional market economy has produced signal results in economic development that cannot be explained away by any amount of ingenuity and mental effort on the part of conservative economists and statisticians. But the same ghost has acted in his typical manner also in these more determined attempts to escape the shortcomings of the free market system. There exists, indeed, an overwhelming amount of evidence from centrally planned economies showing that the active and positive participation of individual enterprises is not released through a system of quantity targets (gross output measured in volume indices) established from above. Nor, is it possible to achieve the desired results by such a simple system of incentives as that of paying a premium to an enterprise according to its overfulfillment of the centrally established quantity targets or according to its ability to increase its volume index of actual gross outpnt over that of the preceding year.

There are several reasons for the failure of such incentive systems. One is that quantity targets, established from above, may induce the enterprise directors to conceal their true production potential. Another is that they do not encourage directors to use imagination and effort in economizing of input elements. A third reason is that these systems do not induce these directors to help rationalize production and realize desirable investment within their specific fields. And fourth, they do not offer the inducement to improve the quality of the products, because the establishing of quantitative targets of goods and services can only, to a small degree, cover the infinite variety of improvements in quality that constitutes a basic element of economic progress.

A few examples will suffice to indicate the nature of the experiences one has had. In the Soviet Union, in the period before 1957, one worked according to what may be called the ministerial system. There was one all-Union central ministry for each group of goods. Because of frequent uncertainties of supply from other ministries, each minister was tempted to set up his own ministerial factory for the component parts he needed. This led, of course, to inefficiencies of various sorts. There were also bureaucratic delays in settling questions due to the scattered locations of enterprises over the whole country. This motivated the abolition in 1957 of the central ministerial system and the introduction of a territorial system. However, this reform only replaced one type of difficulty by another. The wishes and plans of the different regions were difficult to reconcile and make consistent from an all-Union viewpoint. These difficulties encountered in the regional compartmentalization have recently released a desire to revert, at least in part, to principles of a more all-Union character.

As a final example, we may mention the Soviet attempt at circumventing human shortcomings by separating the industrial steering problem from the agricultural steering problem. This attempt, as could have been safely predicted, has been a failure. It constitutes a flagrant violation of the basic condition that a steering system must be comprehensive, i.e. that it must embrace simultaneously all facets of the economy. It is obvious, for instance, that agricultural production depends essentially on agricultural machinery and fertilizers, and both these means of production are industrial products.

The suggestion I have to make regarding ways and means of finally killing the ghost, or at least subduing him to some extent, is not presented, of course, in a naive belief that here is an « open sesame » that will, in one stroke, solve all difficulties. Rather, it is a suggestion as to *a way of thinking* which I believe is a *conditio sine qua non* for real progress in our search for a solution.

We must begin by making a clear-cut and precise distinction between two phases of the steering work: the selection and the implementation.

The selection analysis is a study of what can be obtained or ought to be obtained if only one considers the following: first, such basic conditions for the economic activity as the technological relations and the most deep-rooted relations governing human behaviour, e.g. utility and its effects on demand; and second, the preferences regarding the results to be obtained in the nation as a whole, or in the world. In the selection analysis we pay little or no attention to the system of economic institutions under which the economic activity of the nation or that of the world takes place or ought to take place.

The implementation analysis is a study of the kinds of national or international institutions most helpful in bringing about *that particular constellation* of the national or world economy which has emerged as the optimal one in the selection analysis, or at least to bring about a close approximation to that constellation.

The selection analysis must precede the implementation analysis. If we go about it the other way, we would be putting the cart before the horse. The selection analysis must be built on a quantitative *decision model,* as distinct from an explanatory model or a forecasting model. This will, I believe, be a distinctive feature of the econometric planning work of the future, since our main concern will be research work on how the economy can best be steered.

In the technical part — which I shall not discuss in detail here — a serious warning regarding a very popular « planning procedure » is in order. It is a procedure that owes its popularity more to its simplicity than to its real relevance for true planning.

I am referring to the popular procedure of initially guessing at a « reasonable » national growth rate that « could probably be obtained », and from this assumption drawing conclusions regarding the production needed in special sectors of the economy, the size of needed investments, etc. The reason why this method has become so widely used is to be found, I think, in its simplicity rather than in the fact that it is realistic and rational.

The special aspects of the economy, such as production in the various sectors, the size of investments, etc. are, in fact, *not determined* even if the national rate of economic growth is given. There may be many different development patterns that all give the same rate of growth of GNP (The Gross National Product) or of some other statistical measure of which one may think.

Nor have we any assurance that the growth rate guessed at is the *optimal* one, i.e. the best growth rate obtainable when the structure of the economy, as well as the special preferences which have been put up for the course of the economy, are given. The optimal growth-rate will only emerge as a consequence of a rational decision analysis which takes as its starting point the preferences and the fundamental data describing the structure of the economy. To *start* by target setting, whether it be a specific figure for the growth rate of GNP or some other specific target, is again to put the cart before the horse.

Certainly, we must *end up* by formulating targets; but before this, there is a long way to go — namely through the entire selection analysis.

NOTE. — The verbatim record of the technical parts of Professor FRISCH's presentation is not given here. Nor are the several mimeographed documents distributed by Professor FRISCH reproduced. This material will subsequently be coordinated by Professor FRISCH and published by him separately.

DISCUSSION

ALLAIS (*)

I said we must be very grateful to Prof. FRISCH for his very clear, and, I would say, provocative exposition of the future of econometrics. I would also say that I have had much occasion to admire Prof. FRISCH's work. From a technical point of view, I completely agree in general with his position, but his paper also expresses many views which rest on value judgments and have evidently many political implications. Professor FRISCH has given his point of view in a very excellent way as far as clarity is concerned, but I cannot follow him insofar as fundamental questions of applied political economy are at issue. His paper raises many questions which are connected with the ordinary work of the econometrician, but which, it must be recognised, have a high content of a political nature.

I am sorry that my knowledge of English is not such as to permit me to express myself with all the nuance of meaning which is desirable and this makes it difficult for me to specify my personal views clearly to you. But I think the purpose of this meeting is to bring out divergences of opinion very clearly and it may perhaps not be without utility to pur forward a different view from that of Prof. FRISCH.

(*) Comments on the FRISCH's paper presented to the Study-Week. Only a small part of this paper has been maintained and Professor ALLAIS has not had the possibility of reading the revised and reduced paper printed on the preceding pages.

1) First, as I already stressed on the first day of this meeting, I think econometrics should remain neutral, i.e. we must avoid introducing political views into our discussions. Personally, I would say that I am a neoliberal, but I think political views should remain outside the technical discussion of econometric problems. I do not accept at all that Prof. FRISCH's paper can be regarded in any way as specifying the main lines of the future of econometrics. Econometrics is a very powerful tool of analysis but nothing more. In itself, it cannot determine what economic policy should be, but only analyze observations and derive, in a rigorous way, the consequences of specified hypotheses. Had Professor FRISCH said in his paper: « I admit as hypotheses, first, that a competitive system cannot realize the « high goals of rapid expansion, growth, and social justice, and second that these goals can be effectively realized in a central planned economy », I would not have said anything because from a scientific point of view, it is always possible and legitimate to make hypotheses aud to discuss the results. But instead Professor FRISCH has spoken of these two hypotheses as if they were well established facts.

Econometrics must remain limited to the discussion of technical questions. Certainly it is possible and admissible to discuss scientifically the consequences of hypothesis of a political nature, but it is necessary to avoid connecting them with political and ethical views and with value judgements.

2) Professor FRISCH spoke of the « simplicity » of the competitive system but, I think, the same judgment can also be made on FRISCH's proposals for realizing justice and rapid growth, looking for example at the first lines of § 5.1 (page 5).

Taking FRISCH's paper as a whole, I would say that things are much more complex than his paper makes them out to be, and I think we must be very cautious about all the statements made. As an illustration of this complexity I will put forward for discussion some very important questions on which it is evident that it is impossible to follow the FRISCH conclusions.

I do not say that FRISCH's ideals should be criticised. On the contrary, I have the same social ideals as Professor FRISCH, but I don't think that his practical proposals would in any way forward their realisation. My purpose is to comment on some postulates which are generally accepted by those favour central planning, but which are really very questionable.

3) First question: what exactly are the social preferences Professor FRISCH spoke about? Should there be a steering committee to decide what they must be? But we must be very cautious here. I think the proposition that some political steering committee should have power to decide the aims to be or pursued by society is in fact a very questionable one. Prof. FRISCH felt that it was necessary to correct this position, and he says in his paper (page 2) that plebiscites could be used.

But, from a democratic point of view, the use of plebiscites is itself very questionable. There are many difficulties which I cannot discuss here in detail. But, for instance, who will have the right to write the text of the referendum? This is very important because the answers can evidently be biased. Many Frenchmen are very sceptical about plebiscites to-day. Personally I consider them a very dangerous procedure and in any case I think it is impossible using plebiscites to decide what national preferences are in fact. The definition of social preferences is a very difficult and complex problem and I think the purposes of the society cannot be decided in a definite way from a national point of view. There are millions of people who have their personal and very legitimate preferences. Certainly, there are many decisions which only a government can take, but these decisions are only one element of what Professor FRISCH calls « social preferences ».

And we must consider not only the central government but public agencies of any kind. It is imposible to reduce the problem of social preferences to the problem of defining one preference function and one preference function only. The problem is much more complex indeed and from this point of view the procedure which Prof. FRISCH has proposed is not at all satisfactory.

4) The interest of the community is referred to in the introductory text of this study week. But what is the interest of the community? Who will decide what the real interest of the community is? Professor FRISCH would answer: the majority. But in a real democracy there are many questions which cannot be decided by majority votes.

Minorities have rights which must be respected. So this question is again very complex, and personally, I cannot see in any community anything other than the superposition of individual interests. And if this position is correct, it would appear to be impossible to replace individual preferences by a single preference function for a whole society.

5) Prof. FRISCH has formulated excellently the stress he puts on the relations which must obtain in all cases between the political authorities and the people responsible for planning and he has said that the political authorities can correct the plan or use it in some way. But I would say that, from a practical point of view, this in quite impossible, because politicians are incompetent in econometrics. Who is to decide what the fundamental variables of the model are? Who will decide if the calculations have been made in the right way or not? All these questions are very difficult indeed. What the political authorities can validly decide is the general rules governing the decisions to be taken. But to the extent that millions of decisions are in question, they cannot be taken by any central agency. And the specific value of a market economy is that it provides a very valuable tool for the organisation of decentralised decisions.

At any event, a political assembly can only discuss questions of principle, and decide the general rules governing the decisions to be taken. It can discuss technical plans and economic calculations neither validly nor efficiently.

6) Professor FRISCH's starting point is that the overall purpose of social policy should be the human personality. I agree completely with this principle, but the question is: what are the different aspects of respect for the person? Again, who will decide what exactly is

respect for the person, the rights of the majority, or the rights of the minority? Whatever the solution to this fundamental problem may be, I do not see it in FRISCH's paper. In any case, majority rule cannot give a valid and acceptable answer to every question.

What exactly is social justice? To speak about social justice is quite appealing in public discussion. Everyone is for social justice. But my own experience has convinced me that social justice is accepted by different people in very different ways, and as far as I can judge, everyone accepts what is in his interest as just, but considers anything contrary to his interest as unjust. Reality, unfortunately, is such we must appreciate that there is no objective concept of « social justice » at all, but only the conflicting interests of millions of people.

My conviction is that FRISCH's paper oversimplifies very complicated and complex questions and that it is only in this way that he can justify the central planning procedure proposed by him. But as he said himself at the beginning of his paper, simplicity can be questioned, and to simplify problems is not to solve them.

7) In FRISCH's paper, everything is derived from the consideration of a single preference function only. But as I have already said, we cannot consider only one preference function. And if we agree that in parallel with the social preference function we must take individual preferences into account in some way, then there is no longer a single preference function, but ten million, a hundred million, and, for the world, three billion preference functions, and from this point of view I cannot see at all how FRISCH's paper could work in reality. The question is much more complex, much more difficult. We econometricians must recognize that it is impossible to reduce the whole problem of social organization to a problem of central planning.

In fact and in my opinion, only a decentralized organisation in an appropriate framework taking parallel account of a market economy and some central decision making by the government in its own sphere can provide a correct solution to this very complex problem.

Only in totalitarian societies can the problem of social and economic organisation be reduced to the formulation of a preference function.

8) Professor FRISCH has said that there are two things which are different, and which require to be treated separately. The first is what he called the selection problem, and the second the implementation problem. In my opinion, this is absolutely impossible. I am an engineer, and I can give you a very good example of such an impossibility. In coal mines we don't know at all what the production functions are exactly. No engineer exists who can specify what the production function is in a concrete situation. What engineers do is to choose between different projects in comparing their discounted net present value (in french: leur valeur nette actualisée). And for this purpose they use a system of prices which only a market economy functioning in an appropriate framework can provide. Thus it is impossible to separate selection and implementation. From this point of view, I cannot see at all how the FRISCH system could work. We don't know the production functions, and correct decisions cannot only be taken in a decentralised system with the help of an appropriate price system.

9) And again, what should be considered as desirable growth? Growth of population, growth of efficiency — is growth really desirable? Some people prefer stability to growth. Personally, I am for efficiency, but that is a personal and subjective view. Other people may prefer the stability of their jobs; they can definitely prefer stability to efficiency. In fact, growth is not such an unquestionable goal as Professor FRISCH was suggesting at the beginning of his exposé.

For me what seems in fact desirable is not growth but simply people's happiness.

10) Professor FRISCH has suggested that the centralized economies have grown faster than the market economies. In fact, at the least this statement is open to question and personally I think that it does not conform to the real facts.

11) Professor FRISCH has made some very strong criticisms of the market economy, and I agree completely with him. At least in some aspects, I recognize that the market economy has many, many drawbacks. But such criticism should not remain limited only to some points. It should be extensive.

Rightly, Professor FRISCH has critised private monopolies, but he did not say anything about public monopolies. He did not say anything about trade union monopolies, the impact of which may be much greater. Personally, I think that the unions play a very useful role but it seems impossible to me to criticise only those deviations which relate to the market economy, without simultaneously analyzing the other deviations.

If we want to be objective, we must compare the drawbacks of the market economies with the drawbacks of the politically and economically centralised systems. One needs only to look at the history of the collectivist society in the U.S.S.R. to be convinced that central planning has some drawbacks which can do more damage to social justice than can those of the market economy. The millions of unemployed in the United States in the thirties can be compared with the millions of dead in the U.S.S.R. during the same period.

Is it scientific to give an idealistic view of the planned economy by comparing it with the reality of the market economy? If we want to compare, we must compare things which are comparable. In other words, if we look at the reality of market economies, we must at the same time look at the reality of the collective and centralised economies. And if we discuss what a collectivist and centralised economy would be ideally, we should discuss what a market economy working in an appropriate institutional framework could be, not what it is. But it is not fair to compare the real aspects of a free economy with the ideal aspects of a collective and centralised economy.

It is absolutely scientific to stress the aspects of the market economy, but if so one is faced with the necessity of stressing at the same time what happened in U.S.S.R. in the thirties and in the forties and what has happened in communist China in recent years.

12) If I can add one word about the market economy, I would say one thing, namely that the people I know who are in favour of the market economy are not at all in favour of the market economy as such, as it is. They are in favour of a market economy operating in an appropriate institutional framework, and if we want to do justice to the market economy, I think it is necessary to study the institutions of the framework of the market economy very carefully, before coming to any conclusions.

As I have read in the FRISCH paper, we should not rely on the market economy at all, but my conviction is, on the contrary, that it is impossible to solve the very difficult and complex problems we have to face without some reliance on the market economy.

It is my convinction that with appropriate rules and in an appropriate framework, a market economy can give reasonable participation in the decision-making process to everybody, and to every minority.

13) In conclusion, I greatly admire the scientific work of Professor FRISCH, but I cannot follow him so far as the main themes of his paper are concerned.

For me, FRISCH's paper appears as a long and convincing demonstration of the practical impossibility of planning in FRISCH's sense. Nevertheless, I recognize that this is a very personal and subjective view.

But what appears to me as indisputable is the necessity for the econometricians to remain neutral.

From an objective point of view, it is absolutely impossible to define the econometrics of the future by reference to Professor FRISCH's paper.

In FRISCH's sense there are in reality at least two, three or may be ten econometrics of the future: the STONE future, the WOLD future, the ALLAIS future, and so on.

Thus, in my opinion, it is not desirable to connect econometrics with a social philosophy of any kind whatever respectable it may be. We Econometricians must, as such, remain neutral, we must limit ourselves to the study of econometrics in itself.

[17] *Frisch* - pag. 16

This does not mean that political science is without interest but it does mean that we economists must remain very cautious about what the interest of the community, social justice and so on really are. Excuse me for having been so long and thank you.

DORFMAN

I do not wish to enter into the important and heated discussion between Professor FRISCH and Professor ALLAIS on the role and propriety of social planning, but merely to point out the relationship between Professor FRISCH's paper and my own in two respects.

However one feels about social planning, one must concede that political decisions that influence the development of an economy have to be made. For this purpose one needs some criterion for judging whether social effects are good, or moderately good, or bad and this is what Professor FRISCH's capital F function does. One must have something of this sort or no social decisions can be made on any rational basis.

I do not believe that this social preference function can be ascertained, however, by the method that Professor FRISCH proposes, namely by artfully constructed interviews designed to disclose how people evaluate various possible states of their societies. In general, people do not know how they will make an important decision until they are confronted with it, and they cannot tell you. Therefore I proposed that we attempt to determine social welfare functions by inspecting how people have decided in the past rather than by asking them how they would decide in the future. My purpose, however, was the same as FRISCH's: to determine a scale of social values.

There is another significant divergence between us. Professor FRISCH distinguishes sharply between the problems of selection and implementation. But I do not feel that this distinction can be maintained. The preference ordering of two economic policies depends not only on their consequences in terms of rate of economic growth, per capita income, level of employment, and so on, but also on the implementation side of the policies themselves, for example on the extent to which they restrict economic freedom. Thus one can compare only states of the economy that can be attained by instruments that are similar in their social impacts and this implies that constraints have to be brought into the picture from the outset for they describe the states that can be attained by means of a limited set of economic instruments.

KOOPMANS

I am approaching my comment here in the same vein as Prof. DORFMAN. I am not addressing myself to the question whether society is or should be moving in the direction that Prof. FRISCH's paper indicates. Rather, for the purposes of the discussion, I am accepting his assumption that this is the direction, and speaking more technically to the point whether the particular layout and scheme of Prof. FRISCH's ideas is efficiently designed to achieve the purpose that he has in mind. I have really only two points that I would like to raise in this connection. One concerns the very strict separation of the determination of structure from the determination of preference. On this point I find my thinking to be somewhat related to that of Prof. ALLAIS. It would seem to me that the policy maker who is being interviewed in order to obtain a representation of his preferences will be neither able nor willing to be too specific about these preferences as long as he does not know what the implications of his indications are. He is likely to be pragmatic — not only a man who thinks abstractly about his own preferences. To stay in his position he must respond to pressures and perhaps even threats in order to be effective over a period of time. Even a man of great wisdom would still have to be aware of what he is expected to do by a number of groups who have ways of making their desires effective. Therefore if he is presented by an econometrician with questions « what are your preferences » or « what is the form in which you would mould what you regard to be the preferences that should guide this planning », he may feel that he is being tricked even though without such intent. He is not an econometrician, and in fact in order to feel comfortable in the situation in which he is placed, he would have to be not only an econometrician but an electronic computer as well. He would want to know what quantitative consequences his indications of preferences would have in order to feel comfortable about the answers he is giving, I wonder whether a better and a more effective procedure, which would get more effective cooperation, would not be an iterative one. While the policy-maker is being asked to state his preferences, he would be assured that this is for a trial run only and that the outcome of the computation is to be presented only to him or to his associates before these preferences would be considered firm.

My second point has to do with the question of preferences being necessarily temporary and having to evolve by experience. It is stated that the broad purpose of planning is, among other things, to realise the high goals of rapid economic growth and social justice. Of these if would seem to me that social justice is the more permanent one whereas rapid economic growth is a more temporary one.

Finally I have a point which is put more in the nature of a question to Prof. FRISCH. This has to do with the way in which the steering prices would be used in order to steer the economy. If the production system has constant returns to scale it would seem that prices are a poor instrument by which to bring about moderate changes in quantities. The production set is, in a two-dimensional case, the set of all points that are on or below a way out of the origin. Them for certain price ratios profit maximizing quantities are found in the origin. For other slightly different price ratios, profit is the higher the larger the output. So in a strictly linear technology, the response to prices is likely to have a flip-flap character.

Now, one would ask, if the competitive market system that is prevalent in many countries is, in a way, a model for the steering of the economy by prices, why does that flip-flap behaviour not manifest itself so clearly in the competitive market as I am concerned that it might manifest itself in the steering mechanism that we are discussing. I believe that in the competitive market system prices are very effective in the long run as conveyors of information and of incentives, but that in the short run a good deal of quantity information goes back and forth between suppliers and demanders in all markets. My question to Professor FRISCH is whether in his bleuprint for the operation of a planning system he does not need also to provide for circulation of quantity information at all levels, or whether he thinks prices alone will be sufficient.

MAHALANOBIS

I should like to make a few observations at two levels from the point of view of a country like India. First, I share the doubt expressed by Dr. KOOPMANS whether a political leader or an administrator or someone who is responsible for decision-making would be able to understand the implications of econometric choices. Secondly, even if he does understand some thing at a technical level, whether he would be able to influence the political or social decisions in accordance with econometric consideration. And thirdly, whether such a leader would not make mistakes which would have their own consequences.

I believe there is a good deal of validity in the doubt to which Professor KOOPMANS has given expression. I have myself continually faced the type of question asked not only by Professor FRISCH when he was in India several years ago, but also continually since then in connexion with planning. Even when somebody would like, would have felt it advisable, to make a decision one way, he might have to remain silent because of uncertainties of the political consequences. This is a serious difficulty. I am speaking from experience. This is, however, only one level.

At another level, I should very warmly welcome the outlook of Professor FRISCH because I believe this would be of great educative value. I welcome this imaginative approach, not because I think a push-button type of dcision can be achieved immediately or even in a few years but because I believe his outlook and his approach can be very important factors in an educative process, at the decision-making level and at the level of national planning. I myself have seen during the last ten years in India the beneficial effects which are coming out of such efforts, not only on the part of Professor FRISCH and other specialists like him, but also on the part of those who are working on such methods in India. I see a beneficial effect — not directly, but in influencing the thinking of individuals and groups and making them more sophisticated in their outlock.

I also see a danger if such studies are taken superficially or are imitated, as such things are apt to be very often in the underdeveloped countries, due to fashion or because of the high prestige and authority of the advanced countries. A superficial imitation of advanced countries at too early a stage may and have often become the most serious obstacle to progress in the under-developed countries.

ALLAIS

Can I stress some technical points? In a mixed economy there are two sectors: the private sector and the public sector. So, the first question I would raise is: could the state formulate a preference function for the private sector? And if so what would this preference function be? So far as the public sector is concerned, we meet the same difficulty. We must consider not only the state but also regions, cities, public and semi public agencies and so on. Is it possible to take account of these different operators in one single preference function?

I have many doubts about this possibility.

I do not see at all how it could be possible or desirable to represent finally different and probably conflicting views by one and only one preference function.

Would it not be better to allow every operator some purchasing power and to leave him free to use it as he sees fit?

FISHER

I have one or two points to make. Both of these are on issues brought up during the discussion, the one by Professor KOOPMANS and the other by Professor DORFMAN.

Professor KOOPMANS raised the point that small changes in prices might induce large changes in action. This is, of course, a possibility, but it is less serious the more different ways there are of doing things. For example, even if the technology is a linear programming one, the production possibility frontier will approach a continuous surface if there are many different activities which cover the entire nonnegative orthant.

Now the question of discontinuities of this type is also relevant to Professor DORFMAN's remarks and indeed my comment here should be taken as a comment on his paper rather than on FRISCH's. Professor DORFMAN wants to present policy makers with shadow prices when they are at a particular vertex and see what they will do. The discontinuity problem arises in this connection because there will generally be more than one set of shadow prices at a vertex. Each set will be associated with movement from the vertex one is at to another particular one. The alternatives must be presented to the policy maker therefore in a form which insures that once he has said he will move to another vertex he will not then also want to move back at a different but still appropriate set of shadow prices. One must therefore ask questions which bracket the range of admissible shadow prices. Once again, if there are numerous activities, this is not a serious problem because the sets of shadow prices associated with a given vertex will not be very wide.

WOLD

It seems to me that Professor ALLAIS is dramatizing the argument a little. It has not occurred to me that Professor FRISCH nor anybody else believes that it is possible to arrive at something like the actual truth when setting up a utility function for a political decision at the macroeconomic level. The purpose of the device is much more pragmatic, in the direction of clarifying the problem, articulating different political views by specifying alternative targets and alternative instruments, targets and instruments that can be modified from time to time, the limited aim not being solutions of absolute optimality, but rather solutions that make workable compromises. If I understand correctly, Professors DORFMAN and FRISCH both adopt this pragmatic view. Thus it is seen that the pragmatic approach is far from uniform, and that it leaves room for widely divergent views about fundamental problems.

Coming to my second point, I am not altogether happy about the points in Professor FRISCH's papers and his presentation where he limits himself to deterministic approaches. It is always necessary to simplify in model building, and it is also granted that deterministic assumptions are sometimes adequate; however, I feel uncomfortable when Professor FRISCH says that deterministic assumptions are a dash forward — I think it is a dash backward. It is an illusion if you believe that you can get rid of difficulties in economy-wide models by supposing that you are so precise that there are no unexplained residuals in your approach. To reduce the residuals you must stratify in great detail so as to obtain homogeneous cells in your statistical tables; when the cells become small, however, the law of large numbers ceases to work and instead of more deterministic regularity you will run into more randomness and irregularity. The problem of the model builder is to strike a sound balance between the gain in information given by a finer stratification, and the loss in information when the law of large numbers is weakened. In a sense the residuals are the back side of the medal of large numbers; they will however do no harm if they are treated as stochastic variables, and if the model takes them into account in such manner tbat the operative use of the model is in accordance with the mathematical rules for operating with random variables. This last principle takes care of the pitfalls when a deterministic model is stochasticized. Now as far as I can see the deterministic models considered by Professor FRISCH are not in the danger zone in this respect. At least for ordinary input-output models it is known that they can be stochasticized in accordance with this principle (see my report to the Tokyo session 1960 of the International Statistical Institute).

LEONTIEF

In order to understand somewhat better the implications of Professor FRISCH's practical proposals, I would like to ask whether there would be an objection against allowing people to trade goods and services among themselves at what one might call black market prices. In principle at least, the difference between these and the official shadow prices would reflect the discrepancy between the actual shapes of preference function of consumers and their official estimates prepared by the planning authority.

FRISCH

I have about ten pages of notes that I have taken during the discussion. I will try my best to do justice to everybody, but you will understand that this is a very difficult task in view of the long discussion and the complicated points at issue.

One of the Prof. ALLAIS points was that we are discussing the econometrics of the future and in this connection he said that econometrics should be *neutral*. Everything, of course, hinges upon what is meant by neutral. You know that for centuries there has been a tendency to define neutrality in economics by saying that any analysis which takes the free market system as an axiom, is « neutral », but any analysis that has the audacity of questioning the free market system is not « neutral », but « political » and should therefore not be allowed to enter into the ivory tower of the scientist. This has been the situation in economics for a couple of hundred years but this is not the situation any more. Today we have to recognise the fact that there are also *other economic system* that are « in the air » and must be discussed by us as social engineers. I must add that not only are these systems "in the air" today, but I think they will continue to be so. I even think they will constitute the main object of our discussion in the future. If we are discussing the econometrics of the future we have to recognise this, and I will state a personal belief that 100 years from now our grand-children will devote practically all their efforts to the study of those models that deviate from the free market system. They will use only an infinitesimal amount of their energy discussing such things as, say, the stability of the equilibrium in a free market system. This is my conception of the econometrics of the future. A second point that was raised by Prof. ALLAIS was regarding social preferences. He objected to these preferences being decided by the responsable *political* authority. To this I can simply answer that so far there has not been invented any other machinery than the political one, for steering an economy. We have to accept this as a basic datum in our scientific researches.

The purpose of this conference is not to go into a complete discussion of political theory and describe the whole list of political system that are conceivable. But some political system there must be and some authority *has to decide* in the end. This I take as my starting point. I simply accept the existence of a political authority, whatever its nature may be.

A third point mentioned by Prof. ALLAIS: Who can decide on what magnitudes ought to be attributed to the variables? The politician cannot do it because he does not know econometrics. The answer to this question hinges upon the distinction between the gross and the net form of the preference function. Of course the politician does not know the depth of econometrics. This is precisely why the scientific analyst in his interview with the policy maker has to concentrate on the gross form of the preference function, on the « Santa Claus » form of the preference function. From the purely psychological point of view you may, of course, interpret the word gross as anything under the sun. But this is not what I have done. I have put up a well-defined model, I have defined my concept of gross and net. So therefore I must say that I find Prof. ALLAIS' remark in this particular connection absolutely irrelevant to the discussion of my paper.

With regard to Prof. DORFMAN's intervention I believe I got the sense of what he said when he compared my approach with his own. He would rather prefer to take as a starting point some sort of optimal solution, and then subsequently proceed to a discussion with the politicians perhaps using a sort of iterative procedure in this discussion. On this score there is, I think, a very little difference between Prof. DORFMAN's point of view and mine. It is simply a question of how best to shape the interviewing technique.

Prof. DORFMAN also mentioned the possibility of presenting different described alternatives to the politician and letting him choose between them. Possibly, that is not precisely Professor DORFMAN's point of view, but it is certainly the point of view of somebody else around this table. So I think it merits being drawn into our discussion. This is a very natural view point, a very simple one. The idea is that the experts should work out different alternatives. These alternatives should be listed on a big sheet of paper or perhaps each alternative on its own sheet of paper. And then all these sheets of paper should be put on the politicians table and the scientific expert should say: « Now, please, out of these alternatives choose the one you like. »

To me, this is an absolutely impossible procedure and I will explain why. You can use this method if you have a very very small model with two, three or four variables, because then the number of possible alternatives is so small that the situation can be grasped. But if you have a real programming problem with hundreds of variables and thousands of possible alternatives, as you will have for instance in an under-developed country that strives towards industrialization with a long list of investment projects, you will find that the method of listing alternatives can produce nothing but complete confusion. You would simply be facing what an expert mathematical programmer would call *information death*. You would be killed by the amount of information.

You must proceed in another way, you must proceed in such a way that the computing machine takes over the task of keeping trace of all the alternatives. If you are going to do that, there is absolutely no other way to proceed than by starting to define a gross preference function, a preference function in the « Santa Claus » sense. That is the only way to proceed.

This is of course not saying that any solution that comes out of the computing machine is going to be presented to the politician as the sole and final solution. It will certainly be necessary to proceed through a number of iterative steps, with contact between the politician and the scientific expert. I have frequently spoken and written about the need for a continuous cooperation between the politician and the scientific expert. The iterative steps must be in the following sense. You start with a first and tentative formulation of the policy maker's gross preference function. You run the solution and you come back to him and say « Now, this is what I found — is this what you want? — Then the politician will scratch his head and say: « Oh, no this is not really what I had intended. » Then you will have to start a conversation with him trying to find out more precisely what he actually desires. You proceed to a second approximation to his preference function. And so on until you finally arrive at a preference function and the corresponding solution both of which the policy maker can accept.

Prof. KOOPMANS had an important point when he said that the policy maker is not willing to give up his own ideas regarding the structure of the economy. This is — as I said very explicity in my first presentation — precisely the point on which you must concentrate most of your attention in the discussion with the politician. You must make him give up his own ideas, about the structure. Why? Because if he does not do that, you will not be able to help him. It is like a patient coming to a doctor. The patient has some preconceived ideas about how to solve the problem of his sickness. The first thing the doctor must do is to try to get these ideas out of his head. If somebody is suffering from a psychiatric disease and he wants to jump out of the window and kill himself, what am I going to do? I drag him by the neck and give him an injection. That is the first part of the treatment. I am not starting to argue with him at that moment. But when he is quietened down, I start to talk kindly to him and try to explain to him that his own way of solving his problems, namely to jump out of the window, is not a good solution. The solution that is really good for him is of an entirely different nature. And I will try to help him find that solution.

KOOPMANS

There has been a misunderstanding, and I must express myself sufficiently clearly. I was not pleading that a policy maker's own idea of the structure be respected. I was pleading that when a policy maker is asked for his preferences, that he be given a chance to see the implications of confronting those preferences with the econometrician's model of the structure, so that he knows what is implied in the first indication of his preferences — and I think that in your answer to Prof. DORFMAN you have already dealt with my point.

FRISCH

I think this has cleared the question up perfectly. What Prof. KOOPMANS says right now is just what I said a little while ago about the iterative process in discussion with the policy maker. But it is essential that the process be iterative in terms of *complete* optimal solutions. And if you are going to have any complete solution, you must start by a preference function, trying to lead the politician's mind completely away from his preconceived ideas of the structure.

Prof. KOOPMANS had a second point which is really covered, I think, by what we have already discussed. « The whole thing must evolve by experiments ». I think that those were the words used by Prof. KOOPMANS, and of course my answer is absolutely « yes ».

A third point raised by Prof. KOOPMANS was about prices, how they can be used as means of implementations. He draws particular attention to the fact that you may have an economy with flip-flap effects of price changes. This is completely correct. And, as a matter of fact, I think that I have myself in some econometric papers pointed out this fact. In linear models the problems may look very different from what they look in classical production theory. So there is absolutely no difference of opinion on that. But in my system the optimal prices have to be applied to *specific enterprises* and, of course, in any given enterprise you certainly have very definitely an economy of scale, hence a non linear effect.

You have to a large extent a continuous and non linear process. You do not have a constant return to scale *ad infinitum*.

There are quite a number of pages of the paper I circulated, where I speak about non linearity, but unfortunately there was no time to go into it in my oral presentation.

Let me now say a few words about an aspect of non linearity. Suppose you consider a plant that uses either one of two factors of production. You want to influence the director of the plant by formulating incentives in such a way that: 1) The plant produces a certain amount of product; 2) It uses one of the two alternative factors of production rather than the other. Then you may consider using two non-linear incentives. You may define a premium as a non-linear function of the amount produced, and with a sharply-defined maximum at the point corresponding to the quantity which you want to see produced. At the same time you may use an other premium regarding the choice of factors of production. Now this is, of course, a very simple example, but something similar can be used in other and more complex cases. You will readily recognize, however, that to work with a non-linear premium is a rather complex affair. You might perhaps do it in certain cases of very great importance but it is absolutely impossible to do it in all the micro-economic details which you have to face when you want to steer an actual economy. That is why I put so much emphasis on this specific accounting medium which is derived from the system of optimal prices. Such a system of incentive is, of course, linear but it may to a large extent be protected against flip-flap effects for the reasons I mentioned.

Prof. ISARD at this conference always reminded us that we had to think in terms of regional problems, and I am glad that he has always insisted on that. For my own sake I have not disregarded this problem entirely in my previously writings. I have here for distribution some big mimeographed sheets which show a work we have been doing at the Oslo Institute of Economics, on the principles of how to study the interrelations between three countries. This job was done primarily by a Swedish economist, Mr. Tom Kronsjo, who has been working with us for quite some time. He uses my system of notation.

If any of you are particularly interested, I have a few more copies for distribution.

Prof. MAHALANOBIS, I must excuse myself if I have not got your point quite straight. There is quite a distance along this table and misunderstandings may arise. As I understood it your first point was this: the politician will not be able to understand a scientific question. My answer is: of course he will not. If you put up to him a description of some 450 dependent variables and 31 degrees of freedom he will be entirely lost. But I don't suggest that you are to put up to him such a system. You should only put up to him very simple questions, one at a time. That is the basis of the interviewing technique. I am absolutely certain that if you can have a quite and not too rushed conversation with the policy maker, being careful that he understands your questions correctly then very meaningful results will emerge. I am not stating a theoretical hypothesis, but basing my opinion on actual conversations with leading politicians including the Chief of Prof. MAHALANOBIS' country.

Prof. ALLAIS spoke about the difficulties of constructing a preference function.

His main point here was that we have many different preference functions, the preference function of different groups, of different persons and so on. Of course we have, I have, for instance, found in my interviewing of high-ranking officials that the Minister of Agriculture will have different preferences from the Minister of Education or the Minister of Industry. Similarly there will be differences between regions of the country and, perhaps even more important, differences of opinion of what should be done in a *group of countries* that are entering into some from of co-operation. Should some industry be located in country No. 1 in this group; or in country No. 2? Which one of the countries should concentrate on agriculture, etc. There may certainly be differences of opinion in these questions and I think, I have fully realized this problem and I have made at least one attempt to overcome it. This is presented in pages 28 to 37 in the blue pamphlet. I have written nine pages about this particular problem but you will forgive me for not having been able to squeeze this into my oral presentation. I am referring to the section on the construction of the coalition preference function. So my feeling is that I have not neglected the question of differences of desires. But somehow such differences must be ironed out if the economy of the country is to be *steered* and not be left to drift whither the wind blows. It is the task of the political machinery to solve the problem of differences of opinion. We as economists have to accept that solution.

Prof. ALLAIS said he had the feeling that this decision problem is solved in the free market system. My answer is that the optimal solution corresponding to the desires of the political authority can never be realized by the free market system, or more precisely there is a probability of measure *o* that it will be. Why is it so? Because the free market system does not open any possibility of expressing political preferences or preferences of a very *special* sort. There is a great number of special political preferences. I have mentioned some, I could have mentioned many more. The problems are specific and they are great in number. I would like to see somebody sit down and list a number of these specific preferences and then set out to prove that the free market system will realize all of them simultaneously.

Prof. FISHER when he went to the blackboard referred to some specific vertex of the admissible region, and he suggested that we may take this vertex as the starting point for a marginal analysis and discuss the matter of optimality with the policy-maker by starting from this vertex. At least that is what I got out of Prof. FISHER's intervention. This is very much against what I have tried to say.

The politician will not even understand what is an admissible region. You must put up to him some questions that are much simpler. These simpler things are precisely what is included in the interview technique I have suggested.

Prof. WOLD, in the first part of his intervention, was so much in line with my own thinking that there is no need for me to go into that question at all.

Subsequently Prof. WOLD mentioned the deterministic approach as compared to the stochastic approach. As I have said, I am absolutely in agreement with him on the ultimate need for introducing the stochastic viewpoint. WOLD rather had the feeling that we had to introduce the stochastic viewpoint already from the beginning.

Then he said that if we don't do that our analysis won't be a dash forward but rather a dash backwards. My answer is: if we try to introduce the stochastic viewpoint from the beginning in these immense models we are facing an impossible task. There will be no dash at all, whether forward or backward. The minimum factor at this stage is a technique of handling a great number of variables. Regarding the *ultimate* goal there is complete agreement between Prof. WOLD and myself. Ultimately we will need multivariables techniques combined with probability theory.

LEONTIEF mentioned a very important point. He spoke about black markets. There is always a danger of developing black-markets or grey markets. Particularly so if you stick to the traditional money economy with prices that move according to where the wind blows. If there is an excess supply, the prices decline, and if there is an excess demand the prices rise. And that is that. If you use some sort of price regulations, you are bound to encounter a tendency forward grey markets and black markets. My answer then will be this: In the first place, you must remember that I am not speaking about trading prices in the ordinary monetary sense. Certainly you must have money to buy the small everyday things, but this after all is a minor problem. I am speaking about the accounting medium derived from the selectionally optimal prices to be applied to the big questions of influencing the policy of individual plants, and in this context, the questions of black markets and grey markets emerges in a different setting. It is not saying that you will not have some form of tensions. This is implicite in the concept of « steering » as distinct from « letting it drift ». Certainly you must try to constitute your system of incentives in such a way that there is as little tension as possible. But it is impossible to avoid tension all together.

ALLAIS

I apologise for taking the floor again, but it is only for a few minutes, for clarity. In Prof. FRISCH's paper, there were in fact two sort of things, firstly political views and value judgements relating in particular to the market economy and, secondly, technical analysis of certain points. I would not have raised any questions about the first part if Prof. FRISCH had said the view he expressed were personal views. But they were presented in such a way that the uninformed reader might think that these views are indisputable and correspond to well established facts.

I appreciate that Professor FRISCH can give a personal definition of growth, but in fact he says that a free market system cannot ensure the realization of rapid economic growth. In this context the word growth is clearly used in the usual sense and not by reference to another particular definition.

As far as the future of economic planning is concerned, personally I am not against any study of the theory of planning. On the contrary, I am very much interested. I think this type of work can be very useful, but on one very definite condition, that is that this theory be expressed in a very neutral form, without expressing definitive and dogmatic value judgments about what a market economy really is or what a central planned economy really is. These questions are details and I don't wish to make too much of them.

But y put four fundamental questions which are very technical and I apologize if I was not able to express my thinking sufficiently clearly. I think Professor FRISCH has not answered these four fundamental questions in a satisfactory way.

First question: when Prof. FRISCH spoke a few minutes ago of the question of preference functions, he said that he had dealt spoken with it in his paper and mentioned especially pages 36 and 37. I have read them again. If I understand Prof. FRISCH's point of view correctly, what he is proposing in fact is to construct a preference function (page 37) but, again, he esteems it desirable to have only one preference function and I personally consider that this reduction is not possible and not even desirable. If we consider different preference functions, for instance two or three only, although the number has no importance, the nature of the optimum problem does change. In any case, from an ethical point of view, this point is also quite important, because if there is but one preference function, this means that some people will have power to define this preference function according to their own preferences. Many criticisms can then be raised. Thus my first question is: Does Professor FRISCH propose consideration of only one preference function or is he ready to consider a plurality of preference functions without attempting to reduce this plurality to one only?

The fact is that he seems to imply that it is possible to reduce a plurality of preference functions to one only which would be maximised. If this possibility does not exist the paper's entire reasoning is deprived of its foundations.

My second point was that — and perhaps I was not sufficiently clear this morning — in general we don't know the production functions at all and if we don't know them, I cannot see how FRISCH's system can work.

My third point is that it is not possible to separate selection and implementation, because in general we don't know the production functions.

And fourthly, it appears to me that it is impossible to reduce the whole process of decision-taking to a dialogue between politicians and planners. Thus, my question is: Does Professor FRISCH intend to plan the whole economy or only one part of it in the manner he described? In the first case, decentralisation would be impossible and the economic system would be very inefficient.

FRISCH

First point raised by Prof. ALLAIS: It is obvious that my paper presents what I personally think is the most appropriate procedure for constructing a national economic decision model. It represents my personal creed at this stage. I thought that this was so obvious that there was no need for making it more explicit at this conference. Second point. Prof. ALLAIS objected against my use of the term « gross » in connection with the preference functions. I cannot accept this criticism because the sense in which I used the world was precisely defined in mathematical terms, and I always used the term consistently in that sense.

Third Point. Prof. ALLAIS mentioned that different people may have different preference functions (and he might have added that one person may have different preference functions at different points of time). All this is, of course, quite true. This problem is essentially a political issue, and must be handled as such. We have no other way of finally deciding about political issues than to use the political machinery existing in the country in question.

It is not our task at this conference to discuss various political theories and types of political machinery. We as economists simply have to take for granted that somehow the nature of what we designate as the national preference function, is *arrived* at. I have been discussing this before in several connections and this is the last time I shall repeat it at this conference.

THEIL

Several discussants have argued that it is not easy to construct social preference functions and I can agree with this. Nevertheless, I would like to suggest that the exercise be carried out. The reason is not that I believe that within a few years this kind of decision-making will be put into practice. It is that the procedure provides us with a method which enables us to find out which parts of the model have to be improved first and which are of less importance This method proceeds in principle as follows. When we mis-specify the model which underlies the contraints and when the decision maker maximizes the preference function subject to these constraints, his decision will in general deviate from the optimal decision and thus depress the utility level below the attainable maximum. This utility reduction is an appropriate measure for the seriousness of specification errors. For the special case of quadratic preference functions and linear constraints this idea was applied on a fairly large scale in my recent *Optimal Decision Rules for Government and Industry*.

FRISCH

Regarding the point raised by Prof. THEIL, I think there is no serious or fundamental difference of opinion between us. Let me only remind you that I have not described the desires of the politicians, solely by means of the preference function. There are also political bounds coming into the picture. I have explained this rather fully in the paper, but unfortunately I did not have time to insist very much on this in my oral presentation.

THE ECONOMIC FRAMEWORK OF REGIONAL PLANNING

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1. *Development planning* has become an established activity in many countries after World War II and both a practice and a theory have emerged. It is only natural that practical devices were in use before a somewhat integrated theory was available and it also stands to reason that both practice and theory are still changing. Most of the common body of theory which is nowadays accepted by a majority of development planners deals with an *economy subdivided into sectors,* that is industries or activities — depending on the type of analysis used. One of the main reasons for working with such models is no doubt that the proper choice of sectors promising « comparative advantages » in international competition is of paramount importance for the success of any development policy.

Even so, however, another « dimension » in economic development is increasingly requiring attention, namely the spatial or *regional* aspect. Many governments, especially those

(*) Professor JAN TINBERGEN was unable to attend although he had hoped to be one of the participants in the Study-Week up to the last minute. We are happy to include in the present volume his communication which he sent in advance.

of large countries, feel they must pay some attention to the regional distribution of well-being and hence of new investments and a large number of studies and activities are in operation for the preparation of regional development. Since this dimension brings into the realm of planning some other aspects known as physical planning and more heavily depends on some social issues, these studies and activities are carried through by a variety of different experts among which the general economist plays a minor role. Together with the need for very detailed decisions involved all this has led to a situation in which the main economic interdependencies are not as much observed as in macro or sectorial planning. There is a clear need for an economic framework for regional planning, that is for models, methods and procedures of such planning. It is the purpose of this paper to draw a few lines along which it is hoped practical work can be done.

2. The addition of the *dimension of space* or distance to economic analysis signifies a considerable increase in the number of variables and of equations to be included in the models of economic behaviour. Moreover their practical use requires the knowledge of a large number of new « data », that is, coefficients occurring in such models. Many of these data are not readily available. In order nevertheless to arrive at workable models, we must simplify as many less relevant elements as possible: the old art of science building. It goes without saying that various types of simplifications will have to be tried out and compared before some standard approach can be attained. It also stands to reason that one concrete situation will not necessarily require the same type of simplification as another concrete case. We propose to make a number of suggestions meant as challenges for further discussion. These suggestions may be seen as a further elaboration of previous suggestions by the author [4].

3. The first possibility for simplification consists, of course, of a *limitation of the number of sectors and of regions.* This means that the model we are going to discuss must be seen as one indicating the main links only; one, therefore, of a first orientation and a stage only in the complete process of planning. From what follows it will be seen that we will distinguish nevertheless between a few categories of sectors, meaning that we have in mind a number of around ten sectors still, maybe a few more even. As for the number of regions anything between two and five may do. This implies that we do not propose to deal with the micro-economics of regional policy. It remains conceivable, however, that once models will be manageable consisting of much larger numbers of sectors and regions. For the time being the precise delimitation of the regions will not constitute too much of a problem; it is self-evident that as much homogeneity in well-being and sectoral composition or demographic characteristics as possible mnst be striven for; a further criterion will be that within a region transportation costs must be clearly lower than between regions. Important natural barriers to transportation therefore represent natural frontiers between regions. Even with the small number of regions proposed meaningful problems may be approached; well-known examples of two-region problems are those of Italy or of Belgium; examples of three-region countries are Peru and Ecuador; problems of urbanization can be dealt with by the assumption of « regions » consisting of gronps of cities of various size classes and a rural region and so on. We must, however, reserve one region of our model to represent « foreign countries » or the « rest of the world » in order to deal with the phenomena of imports and exports. That then means that in the case of Italy or Belgium, just quoted, we have three « regions » and in those of Peru and Ecuador four. We need not emphasize that for problems concerning individual giant projects a subdivision « ad hoc » of the economy can be chosen.

4. Important simplifications can be obtained by a type of approximation well known from the theory of international trade: the assumption that for some sectors transportation costs can be neglected within a certain area and considered prohibitive outside such an area. This is what the theory of international trade does with regard to the factors of production. We propose to do this for some products. In this train of thought we will speak of a *regional sector* whenever its product cannot be transported from one region to another. We think building, the operation of buildings, retail trade and some more services (education, for instance) are good examples. We may occasionally lump these sectors together or only distinguish two of them, a labour-intensive and a capital-intensive one. The consequence of our assumption is that any increase in the production of other sectors in a given region will imply an increase in production in the regional sectors in the same region.

5. A similar assumption will be made with regard to some other sectors, to be called *national sectors*. Their products cannot be moved between nations, that is, cannot be imported or exported. Examples are government services, energy and inland transportation, the latter by definition. As a first approximation often building materials production may be added. Energy will not be an example wherever interconnected electricity networks exist, as in Europe.

For several of the regional and national sectors we may neglect transportation costs if no transportation actually takes place; for some others it will be very essential to take into account transportation costs. This applies to building materials and probably to some agricultural and mining products.

6. The remaining sectors will be called *international sectors*. All commodities appearing in the balance of payments are necessarily of this category; the inverse is not true, namely, that they must appear in the balance of payments. It is

characteristic for these sectors that their production need not be equal to their disappearance into national final demand and inter-sectoral deliveries. It is by manoeuvring with the productive capacities of these sectors that the economy must try to attain the highest comparative advantages, that is, to maximize national product — maybe within limits set by regional policy. As another simplification the transportation costs of the products of these sectors may be neglected.

7. It seems proper to use the method of *input-output analysis* for the description of the necessary inter-sectoral deliveries. Occasionally choices between alternative production processes may have to precede and for some particularly important cases even to be built in into the model making it a linear programming model. It is characteristic for the subdivision of an economy into regions that there may be relevant differences in production costs of the same commodity between regions. This will translate itself into differences between one or more of the input coefficients for the corresponding sector. This may simplify itself up to the point where production in some sectors is only possible in a limited number of regions, a good example being mining or energy. We do not call such a sector a regional sector, as the reader might suppose; its product will serve many regions.

In some sectors *indivisibilities* may play a relevant role: this is true for irrigation, energy and heavy industries. It may be important for some services including university education. We may follow one of two ways of representing this feature: either we may assume a minimum value for investments to be made in such a sector, or we may assume curvilinear relations such as the well-known .6th power of production determining investment inputs. The latter approach cannot be chosen too often if we do not want to make the system of equations unmanageable.

8. *Transportation problems* may play a crucial role for some national (and some international) products of a relatively heavy nature (agricultural and mining products). Here the interplay between location and costs at other locations may have to be brought in. For these products the sub-division of the economy into regions implies that we introduce prices for the same product in different regions or even that we distinguish between alternative regions of origin for one product. The latter assumption may mean that we have, in each region, a set of price notations for the same good with different origin and at the same time implies the assumption of quality differences and imperfect markets. The latter approach, though seemingly more complicated, may actually be simpler since it permits us to assume a finite elasticity of substitution between products of different origin. It also permits the assumption that the price of a good outside its region of production equals its production cost in its region plus transportation costs to the region of destination.

The alternative assumption of a uniform price in each region for each product implies the use of inequalities among the restrictions, usually mathematically more cumbersome to handle.

9. For the type of commodities just discussed a key must be used for the *distribution*, by the buyers, of total needs *over the conceivable regions of origin*. These needs, to be indicated as disappearance, equal the sum total of final demand in the region (consumption plus investment), possible exports to foreign countries and inputs used in the region by other sectors. This disappearance itself originates from production within the region and imports from other regions (including, possibly, foreign countries). In loose terms, it will be bought wherever it is cheapest, taking into account transportation. More accurately the key of distribution must be formalized and two alternative keys have been discussed elsewhere [1]. The remarks

made in the previous sector may be considered preparatory remarks to the application of one or the other of these keys. Clearly there are more possibilities [3]. Mathematically the simpler key is the one where we assume an imperfect market and finite elasticities of substitution between the imports from alternative regions.

10. The preceding sections are a sketch only of a type of model which may be used as a framework for regional development planning. The vague way in which it has been described illustrates the large number of possibilities of adapting any concrete model to the particular structure of the country it has to serve. Thus, the number of regions and their frontiers, the number and nature of sectors and their distribution over regional, national and international sectors may be so chosen as to approach reality as much as possible. Even so hardly all the coefficients it contains will be available from statistical measurement and some of them will have to be chosen rather arbitrarily. This applies especially to the substitution elasticities for the products whose transportation costs cannot be neglected. Probably it also applies to whatever cases of indivisibilities will be introduced.

Even when the model has been established another choice has to be made: the one of the *type of development policy* one wants to analyse. It is necessary to follow the habits developed in the theory of economic policy and to define the aims and means of such a policy. The *aims* may either be chosen as a set of numerically given targets or as the maximization of some social welfare function. For regional policy the welfare function will depend also on regional variables such as *income per head of the various regions* and the most common policies usually imply a reduction of the income differences. Many specifications are possible. To laymen such criteria as first raising the income per head of the poorest region and after a certain proportion to the next poorest region has been reached raising the income per head of the two poorest, and so on, has a certain appeal. To the mathematician other formulations may be more attractive, such as a welfare function depending on national income and (negatively) on the standard deviation between regional incomes per head; or a *weighted average of the per capita incomes of regions* as a welfare criterion, where the weights may be indicative of the importance attached to raising the particular region's income they refer to. Of course there must also be a dynamic aspect to the welfare function, that is that *future incomes* also influence it. We may take care of that aspect in the macro phase of planning, however, and use regional welfare distribution in the annual plans only.

As for the *means* of development policy we may think of a wide variety again, from very few up to a large number of them. We may also think of intermediary targets temporarily considered as means. It is customary to think of *investments* in a number of sectors as means in this sense; the main problem being in which sectors and in which regions to take them so as to serve the aims of the policy chosen in the best way.

11. For practical purposes development policy planning will often have to be undertaken in successive steps (stages) [5]. We assume that in a macro stage the rate of development and the corresponding volumes of total investment have already been chosen for a series of years. We now concentrate, in another step, on the distribution of this investment volume over sectors and regions. We assume that the welfare criterion is a weighted sum of regional incomes. For the time being we disregard economies of scale. Each *international* sector is eligible for an additional investment of standard size (say one million currency units). Moreover, such investment can be made in any of the R *regions*. After it has been chosen in region r, it will require additional investments in regional and national sectors. The former must take place in the same region r. The latter may again be undertaken in any of the R

regions. This means that each project can be presented in R^2 *alternative versions*. With the aid of our model we can determine the additional incomes it creates in the R regions and hence also the weighted sum of regional incomes representing the increase in welfare. We can also determine the total capital to be invested in the *bunch of projects* consisting of the investment of one million in the international sector plus the investments in the regional and national sectors. The method to be used is that we put equal to zero the investment in all other international sectors. I have proposed to speak of the *semi-input-output method* [1]. Thus the income-capital ratio for all R^2 versions of the project can be found. The version with the highest ratio is the « *best version* ». We may now make a list of the ratios obtainable for the best version of investment in each of the international sectors.

The use we can make of this list is that we select the international sector with the highest ratio and invest all available capital in this sector. With regard to international sectors this will mean *complete specialization*. To be sure there will be also additions to production in regional and national sectors involved.

This complete specialization will be avoided whenever we add *restrictions* to the additional quantities that can be *exported* in any one international sectors. Whenever such a bound has been reached, the second-best international sector gets its turn. Complete specialization may also be avoided if we introduce an element of *decreasing returns*, not yet discussed, bnt realistic in agriculture and mining. It is not to be expected that it will be avoided by the introduction of indivisibilities, or increasing returns. But it may be avoided also if instead of restrictions to exportable quantities we introduce a *price level* of export goods negatively depending on quantities produced [1].

The introduction of non-linear inputs brings in the possibility that the effects on regional income of two projects are *not* *additive*. This calls for much more complicated methods of selection of the best set of projects.

The problem of distributing a given volume of investment over a number of sectors and of regions also takes a different mathematical shape when we adhere to the policy criterion called the layman's (cf. section 10). In the beginning of our selection process we are now only interested in the poorest region's income; that is, our weighted sum takes an extreme form, where all other regions have zero weights. After the poorest region's income has reached a certain level we change the weights. Meanwhile we have already added to other regions' incomes, since as a rule each project bunch will add to the incomes of several regions. With the changed weights our preference for the projects not yet chosen may change. This may make for diversification (that is, non-specialization in the international sectors). We may have to stop our selection of projects at the moment that all available capital has been used. Up to that moment we have not only added, by our choice, to the income of the preferred regions, the poorest, bnt also to other regions' incomes. It is not excluded that this « loss » could have been less if we had chosen another order of selection. We only want to mention this point, but we are not going to treat it.

Finally there are problems created by the *time structure* of the investment program. The selection rules we have discussed will not, as a rule, simultaneously exhaust the capital available in consecutive years. If this phenomenon assumes dangerous dimensions we are confronted with the problem of a variety of scarce factors — not so far discussed in this paper — requiring the introduction of shadow prices for these factors or equivalent methods. It is beyond the scope of this paper to deal with this aspect.

12. A few concluding remarks will be made on the *organization* and *procedure* of regional development policy. These

have to be more sketchy even than the preceding sections. We will concentrate on two aspects, namely the possibilities of *decentralization* of policy decisions and the question of the *time order* in which decisions must be made. Both depend on the mathematical structure of the model describing the relationships between the unknowns, that is, the policy parameters or instrument variables [6]. Decentralization, either with regard to sectors or with regard to regions, will be possible whenever a number of sectoral or regional parameters appear in a set of equations not containing parameters of other sectors and regions and sufficient in number to solve for the unknowns concerned. If it so happens that after a decision on some instruments further instruments occur in the same position, decisions on the latter must be taken after decisions on the former. Since the structure of the equations (or, if we like, the matrix of their coefficients) does not only depend on the relationships between the variables but also on the role given to some of them, it depends on the target and instrument variables, that is, on the *type of policy,* what possibilities for decentralization and what necessities as for time order there are. A simple example may illustrate our point. If targets are set for regional incomes — a case different from the ones so far discussed — the production volume of regional industries may be derived at once, provided we can assume a direct relation between regional income and production of regional industries.

Among the most fascinating and pressing problems of sectoral-cum-regional planning is the one which forms of decentralization — whether sectoral or regional — and what order of decisions is optimal [2]. Models of the class discussed in this paper may teach us about these questions.

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CONCLUSIONS

FINAL STATEMENT

At the conclusion of this Study Week, we should like to thank the President, Members and Chancellor of the Pontifical Academy of Sciences for their initiative in arranging an international discussion of the problems of Econometrics and for their generosity and hospitality in putting at our disposal their exquisite and unique Casina di Pio IV.

In the course of this Study Week, we have had the opportunity to review and discuss many recent developments in the following branches of our subject: macro-economic decision models and development planning; optimal growth models; the problem of uncertainty in development programming; the influence of real capital on the growth of the real national income; fiscal policy and economic growth; regional planning; cost-benefit analysis; statistical tools useful in econometric planning; the foundations of dynamic econometric models in probability theory, and estimation procedures for econometric models.

In addition to discussing the prepared papers, we also considered the paths that future research might profitably follow. Throughout our meetings, many different points of view were expressed and opinions varied considerably on a number of subjects; but general agreement was reached on the following statement and proposals.

Although, as an organised science, Econometrics is barely a generation old, it has already made substantial progress and is attracting more and more attention and interest throughout the world. In the last analysis, this achievement is probably due to the effective learning process which results from combining theories developed by means of mathematics, and thus sharing in the clarity, rigour and power of that subject, with observations of the real world, reduced as far as possible to quantitative terms. Iteration between theory and observation is leading in Econometrics, as in other branches of science, to a systematic body of verifiable knowledge about the real world.

Since the Study Week was concerned with applications of econometric analysis, our thoughts on the desirable directions of future research necessarily ranged over a wide area. They can conveniently be summarised under three heads: the analysis of the economy; economic objectives; and instruments of control.

A number of important areas of analysis were mentioned as particularly deserving of further study; the role of capital accumulation in economic development; the relationships of education and of scientific research to economic growth; the desirability of introducing to a greater extent than hitherto a regional dimension into econometric models so as to connect the economic structure of a nation with that of its constituent regions; and the urgency of developing techniques of quantitative analysis suited to the less advanced areas of the world. Emphasis was also placed on the need for the systematic testing of theory against facts in the construction of explanatory, forecasting and planning models, and on the importance of publishing the results of such tests. (Only in this way can the experience of one country serve as a guide to further studies in other countries.)

Our discussion brought forcefully to our attention the need for both empirical and theoretical analysis of the social objectives of economic development, comparable in purpose and quality with current research into technological conditions and economic relationships. Social objectives cannot be deduced scientifically, but they are data that science must take into account in fostering economic development. We still have unduly primitive tools and methods for ascertaining and describing even approximately the objectives of any country or any group within it. For example, we have no effective way of determining how a community acts when it tries to reconcile two such competing goals as a high rate of future economic growth and a high level of present consumption. Progress in this direction would not only be conducive to better planning for given goals, but would also contribute to a clearer formulation of these goals and to an improved level of political discussion of them. This in turn would lead to a more intelligent and satisfactory selection of such goals.

Our discussions also made clear the need for a better understanding of the capabilities as well as the limitations of various instruments of economic policies which governments can use in the pursuit of their short and long-run goals. Research on the nature of the instruments available has been neglected in favour of research on more narrowly economic problems, such as production functions and market behavior in the private sector. This neglect has led to the adoption of goals that could not be attained by means of the available instruments and to overestimating the effectiveness of some instruments. In short, more research is needed into what governments can and cannot do in trying to foster economic development and stability.

A side of economic development which we feel cannot be overemphasised is the race between increasing productivity and increasing population. Most of the research discussed at the Study Week was concerned with one aspect or another of productivity; yet measures for influencing the rate of population growth may contribute at times even more to human welfare than measures for influencing productivity. Much work, theoretical and empirical, sociological, physiological and economic, is needed on the population problem. Econometricians can contribute especially through theoretical studies on the relationships between economic development and population growth and empirical studies on the effectiveness of various measures for influencing population growth. They can also contribute by assisting in the improvement of vital statistics and other demographic data under the difficult conditions prevalent in under-developed countries.

Econometrics is a powerful tool of scientific analysis. It cannot in itself determine what economic policy should be, but it can work out in a rigorous way the consequences of specific hypotheses and observations. Thus it can help considerably in the successful functioning of an economic system by bringing about a better understanding of the system and an increase in the flow of information available to those who operate it. This is true whatever the political environment in which that system functions and whatever degree of development it has reached. Naturally, continued improvement depends on the general recognition of the new science by society at large and on the additional resources, both moral and material, that may be expected to accompany this recognition.

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NOTE COLLECTIVE FINALE

A la fin de cette Semaine d'Etudes, nous tenons à remercier le Président, les Membres et le Chancelier de l'Académie Pontificale des Sciences d'avoir pris l'initiative d'organiser une discussion internationale sur les problèmes de l'économétrie et d'avoir mis à notre disposition, avec une généreuse hospitalité, leur exquise et unique Casina di Pio IV.

Au cours de cette Semaine d'Etudes, nous avons eu l'occasion d'examiner et de discuter de nombreux développements apportés récemment dans les branches suivantes de notre discipline: modèles macroéconomiques de décision et planification du développement; modèles pour une croissance optimale; problème de l'incertitude dans la programmation du développement; influence du volume du capital sur la croissance du revenu national; politique fiscale et croissance économique; planification régionale; calculs de rentabilité; instruments statistiques utiles pour la planification économétrique; fondements probabilistes des modèles dynamiques et procédures d'estimation dans les modèles économétriques.

Alors que nous discutions les mémoires présentés, nous avons aussi considéré les voies que la recherche future pourrait suivre avec profit. Tout au long de nos séances, des points de vue différents furent exprimés et les opinions variaient considérablement sur de nombreux sujets; mais la déclaration et les propositions suivantes firent l'unanimité.

Bien qu'elle soit tout juste âgée d'une génération en tant que science organisée, l'économétrie a déjà fait des progrès substantiels; elle suscite de plus en plus d'attention et d'intérêt à travers le monde. En dernière analyse, ce succès est probablement dû à l'efficacité du processus de découverte qui consiste à combiner des théories, développées au moyen des mathématiques et partageant la clarté, la rigueur et la puissance de cette science, avec des observations sur le monde réel, traduites autant que possible en termes quantitatifs. Le dialogue entre la théorie et l'observation conduit en économétrie, comme dans les autres disciplines scientifiques, à un ensemble systématiquement organisé de connaissances vérifiables sur les phénomènes réels.

Puisque la Semaine d'Etudes concernait les applications de l'analyse économétrique, nos pensées sur les directions souhaitables des recherches futures ont nécessairement couvert un vaste domaine. Elles peuvent être commodément résumées sous trois grands thèmes: l'analyse de l'économie; les objectifs économiques; les instruments.

Nombre de sujets importants d'analyse ont été mentionnés comme méritant particulièrement des études plus poussées: le rôle de l'accumulation du capital dans le développement économique; les relations de l'enseignement et de la recherche scientifique avec la croissance économique; l'avantage que présenterait l'introduction plus systématique dans les modèles économétriques d'une dimension régionale grâce à laquelle la structure économique d'une nation serait reliée à celle de ses régions constituantes; l'urgence de la mise au point de techniques d'analyse quantitative adaptées aux pays les moins développés du monde. L'accent fut mis également sur l'importance qu'ont des tests systématiques de la théorie par les faits dans la construction de modèles explicatifs, prévisionnels et décisionnels, comme aussi sur la nécessité d'une publication des résultats de ces tests. (De cette manière seulement l'expérience d'un pays peut servir de guide à des études ultérieures menées dans d'autres pays).

La discussion a imposé énergiquement à notre attention le besoin d'une analyse à la fois empirique et théorique des objectifs sociaux du développement économique, analyse comparable dans ses intentions et dans sa qualité aux recherches actuelles concernant les conditions technologiques et les relations économiques. Les objectifs sociaux ne peuvent pas être déduits scientifiquement, mais ils constituent des données que la science doit prendre en compte quand elle veut favoriser le développement économique. Nous avons encore des outils et des méthodes indûment primitifs pour constater et décrire, même de manière approchée, les objectifs de n'importe quel pays ou de n'importe quel groupe à l'intérieur d'un pays. Par exemple, nous n'avons pas de moyen efficace de déterminer comment une communauté agit quand elle essaie de réconcilier deux buts aussi concurrents qu'un rythme élevé de croissance économique future et un haut niveau de consommation présente. Tout progrès dans cette direction non seulement permettrait une meilleure planification à buts donnés, mais encore contribuerait à une formulation plus claire de ces buts et à une amélioration dans la qualité de la discussion politique les concernant. Ceci conduirait alors à un choix plus intelligent et plus satisfaisant des objectifs.

Nos débats ont aussi rendu clair le besoin d'une meilleure compréhension des possibilités comme des limitations des divers instruments de politique économique que les gouvernements peuvent employer dans la poursuite de leurs objectifs à court et à long termes. La recherche sur la nature des instruments disponibles a été négligée au profit de la recherche sur des problèmes plus étroitement économiques tels que les fonctions de production et les caractéristiques des marchès dans le secteur privé. Cette attitude explique à la fois l'adoption de buts qui ne pouvaient pas être atteints au moyen des instruments disponibles et le fait que l'efficacité de certains instruments ait été surestimée. En bref, il convient de consacrer davantage la recherche à l'étude de ce que les gouvernements peuvent et ne peuvent pas faire pour favoriser le développement et la stabilité économiques.

Un aspect du développement économique qu'à notre avis on ne saurait surestimer est la course entre la croissance de la productivité et la croissance de la population. La plupart des recherches examinées à la Semaine d'Etudes traitaient de la productivité sous un aspect ou sous un autre; cependant les mesures destinées à influencer le rythme de croissance de la population peuvent à certains moments contribuer au bien-être de l'humanité plus même que les mesures visant à relever la productivité. Il faut consacrer au problème de la population de nombreux travaux, théoriques et empiriques, sociologiques, physiologiques et économiques. Les économètres peuvent apporter leur contribution en particulier par des études théoriqnes sur les relations qui lient le développement économique à la croissance de la population et par des études empiriques sur l'efficacité de diverses mesures visant à influencer l'accroissement de la population. Ils peuvent aussi aider à l'amélioration des statistiques de l'état civil et des autres données démographiques sous les conditions difficiles qui prédominent dans les pays sous-développés.

L'économétrie est un outil puissant d'analyse scientifique. Elle ne peut pas d'elle-même déterminer ce que la politique économique devrait être, mais elle peut établir d'une manière rigoureuse les conséquences d'hypothèses et d'observations spécifiques. Ainsi, elle peut aider considérablement à l'heureux fonctionnement d'un système économique en apportant une meilleure compréhension de ce système et en accroissant le flux des informations parvenant à ceux qui le conduisent. Ceci est vrai quels que soient l'environnement politique dans lequel le système fonctionne et le degré de développement qu'il a atteint. Naturellement, des progrès continus supposent que la nouvelle science soit reconnue du grand public; ils dépendent des ressources additionnelles, à la fois morales et matérielles, qui devraient normalement accompagner cette reconnaissance.

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