PONTIFICIA ACADEMIA SCIENTIARVM

COMMENTARII

VOL. II N. 13

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PONTIFICIA
ACADEMIA
SCIENTIARVM

COMMENTARII
Vol. II - N. 13
pag. 1-12

THE PHYSICAL INTERPRETATION OF QUANTUM ELECTRODYNAMICS

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SVMMARIVM — In quantistica electrodynamica, status particulis carens non est stabilis, ideoque ipse non est status vacui. In vacui autem statu multae insunt particulae, quae magnopere mutantur, neque computari possunt, cum parum noverimus leges summarum virium (high-energy). Ut quantisticam electrodynamicam secundum physica principia interpretemur, aliqua ratione uti debemus, qua eiusmodi mutationum medias supputemus et huius calculi conclusiones cum experimentorum effectibus comparare possimus. Ad hoc propositum hic ratio quaedam indicatur, quae Heisenbergianam repraesentationem adhibet et idoneo quodam fruitur factorum non commutantium ordine.

INTRODUCTION

In quantum electrodynamics we are concerned with electrons and positrons interacting with the electromagnetic field. When quantum mechanics is applied to this dynamical system, difficulties of convergence arise from the infinite number of degrees of freedom. One obtains divergent integrals for quantities that ought to be finite, and one is not able to make

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Paper presented on April 26th, 1968, during the Plenary Session of the Pontifical Academy of Sciences.

a straight-forward application of the usual rules of quantum mechanics.

For a long time these difficulties prevented any advance being made in the subject. Then it was discovered, by LAMB and others, that definite rules can be set up for discarding the infinities from the equations, leaving finite residues which correspond to physical effects. In this way it was possible to calculate a shift in the spectral lines of hydrogen, the LAMB shift, and an extra magnetic moment for the electron, the anomalous magnetic moment. The results were found to be in excellent agreement with experiment.

Of course a procedure of neglecting infinities is not acceptable in a mathematical theory. One can neglect only quantities that are small. But the agreement with experiment is so good as to lead one to suspect that there is a basis of truth in the procedure. One is then faced with the problem of modifying the procedure to make it logical, while preserving the good results.

It may seem an irregular way to develop a theory - first to have some rather incomprehensible rules that give the correct results, then to try to make the rules into a more logical scheme. But the development of the ordinary quantum mechanics was somewhat on these lines. The original Heisenberg and Schrödinger equations were rather incomprehensible at the time, but it was found that they could be applied successfully to examples. Only gradually did people work out general methods and the general interpretation of the equations. May be it is in general the most practical way for getting a mathematical explanation of the basic principles of nature, first to obtain successful examples, then to proceed from them to a more complete understanding.

Appearance of the Infinities

One can easily set up the Hamiltonian of quantum electrodynamics, choosing it to give the desired field equation. It is of the form

$$H = H_O + H_Q ,$$

where H_O is the energy of the electrons and positrons alone plus the energy of the electromagnetic field alone, and H_Q is the interaction energy. The latter is of the form

$$(1) \qquad H_Q = -e \int A^\mu \, \bar{\Psi} \, \alpha_\mu \, \Psi \, d^3_x .$$

The field quantities A, Ψ, $\bar{\Psi}$ here are expressible, in accordance with the theory of second quantization, in term of operators of creation and destruction of particles. Let the symbols C and D denote creation and destruction operators respectively. Ψ is expressible in terms of destruction of electrons and creation of positrons

$$\Psi = \Sigma \, D_{el} + \Sigma \, C_{pos} ,$$

$\bar{\Psi}$ is expressible in terms of creation of electrons and destructions of positrons

$$\bar{\Psi} = \Sigma \, C_{el} + \Sigma \, D_{pos} .$$

and the transverse part of A is expressible in terms of creation and destruction of photons,

$$A \text{ (trans)} = \Sigma \, C_{ph} + \Sigma \, D_{ph} .$$

We can set up a state with no particles present, represented by the ket-vector $|o\rangle$ satisfying

$$D_{el}\,|o\rangle = 0 \qquad D_{pos}\,|o\rangle = 0 \qquad D_{ph}\,|o\rangle = 0\ . \tag{2}$$

In order to be a physical state it must satisfy also the supplementary conditions

$$(\partial A_\mu / \partial x_\mu)\,|o\rangle = 0\ . \tag{3}$$

These are the conditions that concern the longitudinal part of A. The conditions (2) and (3) together are sufficient to determine the state.

Consider the effect of applying H_Q to this $|o\rangle$, yielding $H_Q|o\rangle$. We see from (1) that H_Q is a sum of terms, each a product of three factors, a C or a D. If there is one D among them, this term will give zero when applied to $|o\rangle$. However, if all three are C's, we get a term $C_{el}\,C_{pos}\,C_{ph}\,|o\rangle$, which does not vanish. It corresponds to an electron, positron and photon having been simultaneously created.

As a result of such processes, the no-particle state does not remain the no-particle state. It is not stationary. It is thus not the vacuum state. We must depart from the usual picture of the vacuum as the state with no particles.

We may apply the transition theory of quantum mechanics to calculate the rate at which the process of simultaneous creation of an electron, positron and photon occurs. We get the result infinity. There is thus no solution of the Schrödinger equation with the initial state $|o\rangle$. Here the theory breaks down.

One should not really be surprised at this break-down from the physical point of view. The infinity comes from high-energy processes. Now with high energies, quantum electrodynamics is not a complete subject. There are other particles

coming into play, and H_Q ought to be modified for the high energies to take into account these other particles. We do not yet know enough about them to say how H_Q should be modified, but the failure of the theory is evidently to be attributed to them.

We must content ourselves with trying to set up a theory of quantum electrodynamics that shall be valid for the lower energies. With our lack of knowledge of how H_Q should be modified for high energies, the best we can do is to cut off the high-energy terms in it.

Cut-offs

The idea of a cut-off is really a very general one in mathematics. Whenever we have to sum a series which is not absolutely convergent, we can get different results by summing in different ways. The different ways of summing correspond to different kinds of cut-off. The cut-off is merely a device for replacing an infinite sum by a finite one, and some such device is always necessary when we do not have absolute convergence. Instead of sums we may work with integrals and the same remarks apply.

In these mathematical examples we choose a method of cut-off and then proceed to make the cut-off more and more remote, to get the limit of the sum or integral to infinity. In a physical theory we might also make the cut-off more and more remote and get a limit. Alternatively, we might find that the cut-off must not be too remote in order to preserve physical sense. This would mean that the theory is only a provisional one and the cut-off is needed because of our ignorance of the true values for the later terms. The restricted cut-off is ugly, but may be unavoidable when we are setting up an incomplete physical theory.

In quantum electrodynamics we cannot make the cut-off infinitely remote, but must make it at some definite energy, g say. This will be the energy at which the other particles begin to play an appreciable role, say a few hundred Mev. Such a cut-off spoils the relativistic invariance of the theory, but this blemish is unavoidable in our present state of knowledge.

We do not know just where g is to be chosen, nor the precise way in which the cut-off is to be applied. Under these conditions, when we proceed to develop the theory, only those calculations will be significant that are insensitive to the cut-off.

The Vacuum State

When we have a cut-off, there will be a solution of the Schrödinger equation starting from $|o>$. But it will be very sensitive to the cut-off, so it would not be a useful thing to work out. The ket-vector $|o>$ will not be approximately constant, so it will not correspond, even approximately, to the vacuum state.

The vacuum must be a stationary state, so it must correspond to a ket $|V>$ satisfying

$$(H_O + H_Q)\,|V> = a\,|V> ,$$

where a is a number. The vacuum is the stationary state of lowest energy, so it corresponds to the solution of this equation with the minimum a. We may suppose it to be expressed as

$$|V> = \Psi_0\,|o> ,$$

where Ψ_0 is a power series in the creation operators C. The vacuum will have many particles present, in a condition of violent fluctuations. The probability of there being various numbers of particles present in particular states is determined by the squares of the moduli of the various coefficients in Ψ_0, in accordance with the standard physical interpretation of quantum mechanics.

For the calculation of Ψ_0 one cannot use a perturbation method, because the interaction energy H_Q is too large. There is no known method for calculating Ψ_0, even approximately. In any case it would not be profitable to do so, because the result would depend strongly on the cut-off.

We cannot calculate the vacuum state, but fortunately this is not necessary from the physical point of view. There is nothing to observe in the vacuum state, and what the physicist is interested in is departures from it. Such departures correspond to kets $K|V>$.

The Heisenberg Picture

The Schrödinger equation is not useful in quantum electrodynamics because its solutions are sensitive to the cut-off. One can do calculations that are insensitive to the cut-off by working with the Heisenberg picture. One then bypasses the difficulties of the vacuum fluctuations.

Each state in the Heisenberg picture corresponds to a constant ket. Thus $|V>$ is constant. In order that $K|V>$ shall represent a state we must have K constant,

$$dK/dt = 0 \ ,$$

$$\text{or} \qquad i\hbar\, \partial K/\partial t + KH - HK = 0 \ , \tag{4}$$

where K is considered as a function of t and of the dynamical variables at time t. Each K satisfying this equation determines

a state. Two K's that give the same $K|V>$ would determine the same state. But as we do not know $|V>$ we cannot use this result in any way and must count all different K's as determining different states. We are thus led to consider that states in quantum electrodynamics are determined by linear operators, which is a drastic departure from the usual idea of states being determined by ket-vectors.

We may use a perturbation method for solving (1), counting H_Q as small. We first make a transformation to the interaction representation, putting

$$K^* = e^{iH_0 t/\hbar} \quad K \; e^{-iH_0 t/\hbar} \; ,$$

which leads to

$$i\hbar \, \partial K^* / \partial t + K^* \, H_Q^* - H_Q^* \, K^* = 0 \; . \tag{5}$$

We then put

$$K^* = K_0^* + K_1^* + K_2^* + \ldots \; ,$$

in which successive terms are of different orders of smallness. We get

$$\partial K_0^* / \partial t = 0 \tag{6}$$

$$i\hbar \, \partial K_n^* / \partial t = H_Q^* \, K_{n-1}^* - K_{n-1}^* \, H_Q^* \quad (n > 0) \; . \tag{7}$$

Equation (6) shows that K_0^* does not involve t explicity. With a given choice of K_0^*, the other terms are determined by the integration of (7).

In the problems of the Lamb shift and the anomalous magnetic moment we are interested in the energy of one electron in a static electric or magnetic field. So we take K_0^* to be the operator of creation of an electron in the static field. We then proceed to calculate K_1^* and K_2^*.

The Normal Order

When we calculate a dynamical variable such as K^*, involving non-commuting quantities, the result has a different appearance according to the order in which we put the non-commuting factors in a term. In order to be able to pick out definite numerical coefficients and apply some sort of physical interpretation, we must have some standard process for ordering the factors.

Let us ignore the dynamical variables associated with the longitudinal components of A, as they would just complicate the argument without changing anything essential. Our dynamical variables are then the C's and D's. It is found that, to get the correct results for the Lamb shift and anomalous magnetic moment, we must order the factors in each term so as to have all the C's to the left of all the D's. This is called *normal ordering*. When one adopts normal ordering in a solution of the Heisenberg equations of motion, the details of the calculation have a close similarity to the corresponding calculation in the Schrödinger picture with neglect of infinities, and this similarity leads to the two methods of calculation giving the same results for the Lamb shift and anomalous magnetic moment.

With normal ordering applied to K_2^*, one of the terms in it consists of the original creation operator K_0^* multiplied by a coefficient. This term corresponds to a change in energy of the created electron and its coefficient gives directly the

Lamb shift or anomalous magnetic moment. The result does not depend on the cut-off, although some cut-off, not too high, is needed so that the rest of the calculation may be sensible.

These calculations in the Heisenberg picture all follow from the standard procedures of quantum mechanics except for one new assumption, the assumption of normal ordering. This assumption is essential for getting results, and we must try to understand it physically.

Physical Interpretation

In the Heisenberg picture the ket $|o\rangle$ varies with the time, so we should write it $|o_t\rangle$. It is determined by the equation $D_t|o_t\rangle = 0$, where D_t denotes a Heisenberg destruction operator at time t.

The normal ordering comes into play for a dynamical variable K if we multiply it into $|o_t\rangle$, because then the only surviving terms in $K|o_t\rangle$ come from terms in K that do not contain any destruction operators. They will be of the form

$$K|o_t\rangle = \Psi_1(C)|o_t\rangle ,$$

$\Psi_1(C)$ being some power series in the creation operators.

If we followed the accepted principles of quantum mechanics we would multiply K into $|V\rangle$, not $|o_t\rangle$, to get

$$K|V\rangle = K\,\Psi_0|o_t\rangle = \Psi_2(C)|o_t\rangle ,$$

where Ψ_2 is another power series in the creation operators. We would then interpret the squares of the moduli of the

coefficients in Ψ_2 as giving the probabilities for the existence of various particles at time t. These would be the particles in existence for the vacuum state, determined by Ψ_0, with some extra particles and some deficiencies. But this interpretation would not make use of the normal ordering of K, and we cannot use it anyway because we do not know Ψ_0.

We find we are forced to multiply K into $|o_t>$. We may look upon $|o_t>$ as the part of $|V>$ without the vacuum fluctuations. We may say that, in multiplying K into $|o_t>$, we are ignoring or smoothing out the vacuum fluctuations — a procedure which we have to follow when we do not know the vacuum fluctuations.

It would be natural now to go the whole way in using $K|o_t>$ instead of $K|V>$, and to interpret the squares of the moduli of the coefficients in Ψ_1 as giving the probabilities for the existence of various particles at time t, in accordance with the usual rules that one would apply with Ψ_2 in the standard interpretation. We get in this way a general physical interpretation for quantum electrodynamics, and possibly also for other quantum field theories, which is somewhat different from the standard one. We may call it the *primitive interpretation*, as it simplifies the situation by smoothing out the vacuum fluctuations.

There is one new feature that has to be taken into account, namely, if $K|o_t>$ is initially normalized, it does not remain normalized. One would have to normalize it separately for each time at which one wants to calculate probabilities.

Further Development

If one accepts the primitive interpretation, there are still some difficulties remaining in the development of quantum electrodynamics. One may apply the perturbation method of solution of the Heisenberg equations of motion taking for K_0*

the operator of creation of a photon. One then finds that in K_2^*, after normal ordering, there is a term implying a rest-mass for the photon. Furthermore, this term is large, of the order g^2.

Of course the photon has no rest-mass. It is therefore necessary to modify the Hamiltonian, introducing into it a counter term to get rid of the photon rest-mass. This counter term is then of order g^2, so it cannot be treated as a perturbation and it disturbs the whole perturbation technique.

We have here an unsolved difficulty, which is probably the most pressing one remaining in the logical development of quantum electrodynamics. As it comes into play for quite small energies, it probably requires some rather fundamental alteration in the theory.