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ON THE FUNDAMENTAL THEOREM FOR POINT-POINT CORRESPONDENCES WITH VALENCY ON AN ALGEBRAIC SURFACE (*)

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SVMMARIVM. — Auctor exemplum exhibet, quo correspondentia algebrica ∞^2, non reducibilis neque degener, super algebricam superficiem, valentiam habeat quae non sit nulla (iuxta verborum notionem a SEVERI vel ab ALBANESE constitutam), cum inversa corrispondentia non eamdem sed aliam valentiam habeat. Id (quod rarissime accidit, si valentia iuxta Severianam notionem accipiatur) tunc tantum evenire potest, cum geometricum superficiei genus est nullum.

The fundamental theorem that if we have a correspondence of valency v on an algebraic curve, then the inverse correspondence is also of valency v is well known. For point-point (∞^2) correspondences between surfaces the corresponding result is less simple. It has certainly been shown [**1,7**] that for irreducible non-degenerate correspondences of valency zero in Albanese's sense (and hence also for correspondences in Severi's sense), the inverse correspondence is also of valency zero. The extension of this theorem to correspondences of arbitrary valency v contains however an important gap, and it is the purpose of this note to demonstrate by an example that the theorem is false even for irreducible and non-degenerate correspondences. The general theoretical grounds for disbelieving the fundamental theorem, which lead in the most natural manner to the example which follows, will not be discussed here: they derive partly from a result of HODGE on the existence of algebraic correspondences [**2**] and partly from the author's work on the intersections of point-curve correspondences on an algebraic surface, which is to be published in a separate paper [**4**]. In

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that paper we shall show that under various conditions the intersection of two point-curve correspondences is a point-point correspondence with valencies in each direction in Albanese's sense, and that these valencies may well be distinct. In particular, the intersection of two point-curve correspondences with valency (in the sense of the author's paper [**3**]) is always a correspondence with Albanese valencies. (It is perhaps worth remarking here that for point-curve correspondences with valency the fundamental theorem holds).

The author is greatly indebted to Prof. B. Segre for some informal discussion on this subject which led to the production of the particular example we give here, and also to Prof. F. Severi and Dr. J. A. Todd for their interest in, and comment on, this work.

The fundamental theorem that, if T is a point-point correspondence of valency v on a surface F so also is T^{-1}, is usually derived from the special case $v=0$ in some such manner as the following.

> T is of valency v, and therefore, if I is the identical correspondence, $T+vI$ is of valency zero. Hence its inverse $(T+vI)^{-1}=T^{-1}+vI$ is also of valency zero, so that T^{-1} is of valency v.

Now for correspondences in Albanese's sense the theorem for $v=0$ is true, as we have remarked above, only if the correspondence is irreducible and non-degenerate. $T+vI$ is, of course, not an irreducible correspondence, nor can we be sure that the homology class on the Riemannian of the product $F\times F'$ (where F' is a copy of F) to which it belongs contains some irreducible and non-degenerate correspondence. (If it did the argument above could be made valid). For correspondences in Severi's sense the difficulties appear to lie very much deeper, since the proof of the theorem for $v=0$ [**5**] does not appear to require that the correspondence be irreducible. Special difficulties arise in the case of surfaces with geometric genus zero (i.e. without transcendental 2-cycles), since on such surfaces, and only on such surfaces, the two definitions of valency are apparently not essentially distinct.

The example we consider is as follows. F is a ruled surface of order $n>3$ and genus q in ordinary space, which is a generic projection of a non-singular ruled surface F^* in higher space. We shall consider correspondences on F, although to avoid any apparent difficulty caused by the singular locus on F, we can imagine the discussion to apply throughout, not to points of F, but to the points of F* which correspond to them. Let P be a plane section of F and G a generator.

Consider the point-curve correspondence T_1 on F in which every point on F corresponds to the generator through it. T_1 is then a symmetrical correspondence, which we call the *generator correspondence,* and, as we shall show in [**4**], it is a valency correspondence in the sense of [**3**]. We have, x being a generic point of F,

$$T_1(x) = T_1^{-1}(x) \sim G$$

(the = representing ordinary equality and the $\sim$ algebraic or topological equivalence).

Now consider the point-curve correspondence T_2 defined as follows. $T_2(x)$ consists of the intersection of F, residual to the generator through x, with the polar quadric of x. $T_2^{-1}(x)$ will consist therefore of the intersection of F, residual to the generator, with the second polar of x.

$$\text{Thus} \quad T_2(x) \sim 2P - G \ , \ T_2^{-1}(x) \sim (n-2)P - G.$$

Let now $\overline{S}$ be the point-point correspondence on F defined by the *intersection* of T_1 and T_2. The indices of $\overline{S}$ are respectively $[T_1(x) \cdot T_2^{-1}(x)] = n\text{-}2$ and $[T_1(x) \cdot T_2(x)] = 2$. $\overline{S}$ is a reducible correspondence since it contains the identical correspondence I simply as a part. [For $T_1(x)$ and $T_2(x)$ intersect simply at x]. Consider however the correspondence $S = \overline{S} - I$. This is of indices $(n\text{-}3, 1)$. We remark that S is irreducible (since one index is unity) and it is not degenerate. We assert that S *is of valency* -1 *and* S^{-1} *of valency* $3-n$ *in the senses of both Albanese and Severi: these valencies are certainly unequal if* $n \neq 4$.

The explanation of the valencies is very easy. A complete continuous system $\}D\{$ of curves of F consists of the sets of q generators. Now for any generator G, $S(G) = G$ and $S^{-1}(G) = (n-3)G$.

Hence

$$S(G) - G = 0 \quad , \quad S^{-1}(G) + (3-n)\,G = 0$$

or

$$S(D) - D = 0 \quad , \quad S^{-1}(D) + (3-n)\,D = 0$$

Thus for any curve D of }D{, S(D) — D belongs to a fixed (null) linear system, so that S is of Albanese valency -1. In the same way S^{-1} is of Albanese valency $3-n$.

That the valencies are also valencies in Severi's sense is immediate. $S(x)-x$ is a virtual set of points equivalent, on the generator through x, to the zero set, and thus belongs to the null series of equivalence on F. Thus S is also of Severi valency -1, and similarly S^{-1} is of Severi valency $3-n$.

Thus this very simple example, which springs naturally from our general theory, is apparently a gegenbeispiel to the Fundamental Theorem. In fact it is clear that if we have a correspondence on F which is made up of (α, β) correspondences on every generator, irreducible on the generic generator, and with $\alpha \neq \beta$ then we must necessarily have an ∞^2 correspondence with unequal valencies $-\beta, -\alpha$. The following simple method of obtaining such correspondences is due to J. A. Todd and F. Severi.

Let C be a non singular curve of genus $p>0$ lying in $[r]$. Consider the Segre variety $[r]\times[1]$ containing $\infty^1 [r]$'s and ∞^r lines mapping these projectively on each other. Let F be the ruled surface formed by the generating lines of the Segre variety which meet C. Then the $\infty^1 [r]$'s cut out on F a linear pencil of curves of genus p (free from base points). Consider an (α, β) correspondence U between the $[r]$'s of $[r]\times[1]$ with $\alpha \neq \beta$. Let T be the point-curve correspondence on F which associates with any point x of F the β curves C cut out by the $[r]$'s which correspond in U to the $[r]$ through x. The intersection of T with the generator correspondence clearly has unequal valencies.

It is perhaps worth remarking that all our examples provide correspondences with negative valencies, and it may well be asked whether it is possible to find correspondences with unequal positive Albanese valencies. Such a correspondence can be obtained as the product of a correspondence with unequal negative valencies and one with equal positive valencies. It is easily seen, by considering the effect of the correspondence on the transcendental 2-cycles, that a correspondence and its inverse cannot have distinct Severi valencies, except on surfaces with geometric genus zero [**5,6**].

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