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MATHEMATICS, SCIENCE, AND THE MATHEMATICAL SCIENCES

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INTRODUCTION

The objective of scientific research is understanding nature, including man, while the objective of mathematics, as a scientific endeavour, is to provide a way of thinking about nature and a means of gaining knowledge. If we accept the meaning of knowledge to be understanding, in contrast to knowledge as simply memorizing information, then it must essentially require mathematics. On the other hand, mathematics can be a form of art, for it is the product of an intrinsic characteristic of the human brain which we call creativity and is motivated by the same emotional processes which produce any other form of art. Think of the harmony of geometrical figures, or the magic of numbers.

The evolution of mathematics as a science implies the expansion of our capability to deal with the complex logical systems needed for understanding the dynamical structure of complex natural systems. During this evolutionary process, mathematics has produced a number of tools and methods, from algebraic operations and geometrical representations to electronic computers, which have increased the power of our mind. Memorizing information with a rapid access rate, handling the data with fast processing systems, and performing long chains of logical exploitation of the brain's capability. In a new form of symbiosis, the man-computer interaction has produced miracles.

Pure mathematical research is both philosophical and scientific when its objective is human thought, human logic, and human understanding. In fact, before the scientific revolution, philosophy and science were considered the same endeavour, and mathematics was a major part of it.

The mathematician is also both a scientist and an artist when imagination and creativity are exploited for modelling the essence of natural structures and natural phenomena.

The very difficult — and very imaginative art — of modelling requires building geometrical and analytical structures which contain only the most essential information. The most important discoveries of the past have resulted from this selection process, which implies both imagination and intuition.

To be successful in the art of modelling one must follow Jacques Hadamard's advice: "It is important, for he who wants to discover, not to confine himself to one chapter of science, but to keep in touch with various others". I would add only that it is difficult to be a successful mathematician knowing only mathematics.

The scientific revolution has been initiated and developed by a strong interaction between mathematics and physics. Unfortunately, in the present century, the strong link between the two disciplines seems to have ended in a divorce.

Many physicists think the mathematicians too often "go off on a tangent", that is, they fall in love with mathematical problems which have no physical significance.

On the other hand, most mathematicians have developed a deep contempt toward the semi-empirical way of dealing with delicate mathematical tools by physicists. In their ivory towers, the mathematicians tend to speak a language incomprehensible

to anybody who does not belong to their "club". To justify this behavior, they emphasize the role that mathematical theories, developed independently of any specific requirement, have played in the development of modern science.

In my opinion, both mathematicians and physicists are wrong. However, one thing is certain: the lack of communication does no good to anyone.

But, I have been speaking too long in generalities. Let me examine a few important events in the history of modern science to illustrate my point.

Let us explore how the interactions between scientists of different disciplines, as well as different attitudes of scientists toward the same discipline, have influenced, both positively and negatively, the development of human knowledge.

More important, let us see how the expansion and evolution of the mathematical sciences has increased our efficiency in solving both old and new problems. The advance of mathematics has and will lead to the development of new concepts and new ideas that have been and will be applied to endeavors in unexpected fields such as the life sciences and the study of human behavior.

1 - Classical and relativistic mechanics

Newton begins the third book of his *Principia* with the dramatic sentence: "It remains that, from the same principles, I now demonstrate the frame of the system of the world". The principles to which he refers are the laws of gravitational dynamics. These laws were the end product of a long, complex mental process, in which mathematics, aesthetics, religion and faith in the perfection of nature as mixed together in the work of Copernicus and Kepler played almost the same role as the scientific methodology and the experimental observations of Tycho and Galileo.

From a few basic principles Newton, Euler, Lagrange, and Hamilton derived the powerful mathematical theory of analytical mechanics. Beginning with differential and integral calculus from the extremal properties of dynamical integrals, the calculus of variation was produced. Later, in Euler's work on geodesic motion, Gauss found the motivation for the foundation of differential geometry. From a generalization of Hamilton, and Jacobi's formulation of dynamics, Cartan developed the idea which led to group transformation. Finally, Liouville, Poincaré, and Birkhoff all working on the behavior of orbits in a Hamiltonian dynamical system, arrived at topology. Indeed, all the modern mathematics developed through the middle of the 19th century, as well as a large part of the development thereafter, was the product of a deep analysis of Newtonian physics. Our understanding of that part of nature described by Newton's laws actually became possible because of the development of new mathematical tools, that in a feedback process, was motivated by the need for a deeper understanding of the laws themselves.

In the day September 23, 1846 classical mechanics reached the most spectacular triumph with the discovery of Neptune by Galle. The very existence and the orbital characteristics of the till then unknown planet were predicted by Adams and Le Verrier for explaining the difference between the observed position of Uranus and those computed on the basis of Newtonian mechanics.

By the end of the 19th Century, mathematicians were primarily concerned with the theory of complex variables, with analytical number theory, and with algebraic forms and invariants. They also developed special taste and aesthetic standards: the initial sign of a developing attitude toward other sciences that would eventually lead to the divorce.

The physicists who wanted to move forward felt, to a certain degree, the need for freedom from the rigorous conditions imposed by mathematics. However, in the 19th Century, inde-

pendent of any demand from physicists, non-Euclidean geometry developed naturally into Riemannian geometry, thus leading to the most spectacular exploit of mathematics.

Albert Einstein built his theory of gravitation, ("a leap in the darkness") on the non-Euclidean or Riemannian geometry of curved, four-dimensional space identified with the physical concept of space-time. In this way, physical laws became incorporated into the geometry of that space. Like Kepler, Einstein was not motivated by any need to explain observations, but merely by an aesthetic sense. Only later was the theory confirmed by experimental evidence: the non-Newtonian component of the precession of the apsidal line of Mercury, the gravitational red shift, and the bending of light beams by gravity.

More recently, the prediction based on Einstein theory of decrease in the propagation velocity of electromagnetic waves due to grazing the sun has been confirmed to within a few parts in a thousand. Mathematics has returned the gift physics made to mathematics by triggering the explosion of the modern mathematical thought.

What will happen in the future? Will Einstein's General Theory of Relativity break down because of new experimental evidence? Or, will it be replaced by a more general physical theory explaining the structure and dynamics of all the Universe—a unitarian, aesthetic view from micro to macro scale, from nuclear interaction to the interaction of galaxies?

2 - Maxwell's equations, special relativity, the ionosphere, and Missed Opportunities

In January 1972, Freeman J. Dyson was invited to give the Gibbs Lecture. The title of his lecture appealing: "Missed Opportunities".

In discussing the "divorce" between mathematics and physics, he mentioned the remark of the physicist Res Jost: "As usual in such affairs, one of the two parties has clearly got the worst of it".

In Dyson's opinion, while in the last twenty years mathematics had experienced a golden age of luxuriant growth, theoretical physics had become a little shabby and peevish. I agree with the second part of Dyson's remarks. But I am not sure I agree with the first one. With divorce, both fields lost a lot. And, luxuriant growth does not necessarily imply quantitative improvement. Any animal may grow luxuriantly fat, without expanding its efficiency.

In fact, even during the really luxuriant era of mathematics in the late 19th Century, the mathematicians lost the opportunity to discover restricted relativity.

In 1861, Maxwell discovered the laws of electromagnetism and wrote the corresponding equations, which expressed the principles in terms of a tensor field expanding through space and time, while obeying coupled partial differential equations of peculiar symmetry.

Almost fifty years later and three years after Einstein's discovery of restricted relativity, Minkowski pointed out the missed opportunity of an extraordinary triumph of mathematics. Maxwell equations are not invariant under the Galilean group of transformations; they are invariant under the Lorentz group, which, from a purely mathematical point of view, has a simpler structure. On this mathematical basis, mathematicians could have been, but were not, able to make independent assessment of the superiority of Maxwell's theory. In fact, physicists continued to consider it a purely speculative hypothesis until Hertz demonstrated the existence of radio waves in 1885.

One other episode should be mentioned in the field of electromagnetic wave propagation. I refer to an opportunity lost by Poincaré, or, at least an opportunity to be more cautious. At the end of the 19th Century, Marconi requested financial support to conduct experiments in the propagation of radio waves between two very distant points on the earth's surface

(Transcontinental communication). Because the presence of the ionosphere could not be reasonably predicted on the basis of actual knowledge at that time, Poincaré rejected the request. He justified his negative attitude by a careful mathematical analysis of the diffraction of radio waves around the earth. Fortunately, Marconi found the strength to pursue the battle and his subsequent experiment produced one of the most important discoveries of the last century.

It appears that this episode was not known to J.T. Schwartz when he gave a lecture at Stanford in 1960 on "The pernicious influence of mathematics on science". A short comment on this lecture may be in order here. In discussing "the inability of computers to be guided by any large context", Schwartz pointed out that mathematics like computers, is characterized "though to a lesser extent", by single and simple mindedness. In Schwartz's view, any jump of imagination outside the boundary of the logical process, (not to mention the criticism and sometime skepticism toward assumptions about observed reality which is fundamental to most physicists) seems as foreign to mathematicians as it is to computers. I do not think this is true for all mathematicians and even if there is some general truth in Schwartz's opinion, when he refers to the particular case of mathematics applied to the social sciences, he has overly exaggerated the situation. In particular he totally dismisses any relevance of the ingenious Birkhoff theorem to statistical mechanics.

3 - Volterra and von Neumann: application of mathematics to the life sciences

In the field of the life sciences, two episodes come to mind. The first is linked to the name of the mathematician Vito Volterra. In the 1920's together with the biologist D'Ancona, he studied the variation in population of two species of fish in a single lake. He assumed that the first specie sustained itself by eating the second, while the second lived off the other, theoretically unlimited, resources of the environment. He found that in the steady state, the two populations varied periodically in opposition of phase. The Volterra model, although simple, explained the mechanism of the observed oscillations in the population of fish. I believe this is the first application of mathematics to a non-physical problem: a problem which has some relation to social dynamics. A similar problem (foxes and rabbits) was considered fifty years later by Kemmeny in his book "Mathematical models in social science". Obviously, an oscillatory character may manifest itself in many other social situations, such as a changing student population at a specific university, or the general evolution of a nation's economy.

The second episode is more recent. In the middle of this century, von Neumann was working on a general logical theory of automata. This is, in fact, the title of a paper reprinted in the fifth volume of his collected works. The main theme of the paper is an abstract analysis of the structure of an automaton with sufficient complexity to be able to reproduce itself. The four essential components of such a system are identified by von Neumann as a "factory" (A), a "duplicator" (B), a "controller" (C), and a "written instruction" (D) for telling A to manufacture the combined system, A+B+C. When C is given an instruction, it passes to B for duplication, then passes to A for action, and, finally, it supplies the copied instruction to the output of A, keeping the original for itself. Von Neumann showed that a structure of this kind was necessary — and sufficient — for a self-reproducing automaton. Von Neumann's conjecture that the same type of automaton structure should also exist in living cells was confirmed by Crick and Watson's discovery, five years later, of the double helix of DNA.

Türing further developed von Neumann's idea of a universal automaton by showing that at a certain level one does not need to make the structure of the automaton more complex to obtain

a more developed system, but only to give it more sophisticated and complex instructions.

Türing, following von Neumann, thought that this basic mathematical result contained the principle of continuous biological evolution. In short, one does not need more complex biological processes for a indefinite evolution. Although very simplicistic, this idea is certainly of basic importance to anyone developing a mathematical theory of biological evolution.

4 - The advent of computers

The advent of computers, a true product of mathematics and electronics, has created a revolution not only in science and technology, but in the very organization of society.

The computer's main characteristics is its ability to process large amounts of data rapidly and to perform long chains of numerical and algebraic operations. I'll give here only a few examples of the impact computers have had on scientific research.

The first and most obvious is the fast processing of huge amounts of data and the extraction of specific information embedded in very large samples affected by noise. Radar Astronomy is a typical example. In this technique, a sequence of pulses of microwave radiation is sent from a large antenna toward a near planet and the echo, or reflected signal, is collected by the same, or another, antenna. The echo of the coherent monocromatic pulse, a short segment in the time-frequency domain reflected from the planet surface, is spread in both dimensions of the same domain and the returned signal carries information on the geometrical and optical characteristics of the reflecting surface as well as on the position and motion of the planet. The strength of this signal with respect to the background noise is so weak that only by processing a large number of returned echoes can the signal be amplified enough to extract the information. Yet, by this technique, the mapping of the mysterious surface of the cloud-covered planet Venus has been possible. A similar, if even more sophisticated, data processing technique makes possible a continum survey of the crustal motion (that is the relative position of a set of points on the earth's surface each separated by several thousand km) with an accuracy of few centimeters.

The second large impact of computers on scientific research is the tremendous increase in capability to solve numerically initial and boundary value problems for systems of both ordinary and partial differential equations. In particular, strong non-linearities like those of the Navier-Stokes equations for large Reynolds numbers may be handled with the computers. We soon hope to handle the problem of instability in boundary layers which leads to regular vorticity and to many other natural phenomena with such regular patterns as to be of deterministic character, e.g. the ripples along a seashore, the sand dunes of the ocean floor and deserts, and many cloud formations. The geometry and dimensions of fish scales or the distribution of feathers on the wing of a bird may also be manifestations of evolutionary optimatization of the same natural process.

These are typical cases where the mathematical model is too complex to be handled analytically. So far, mathematicians have been unable to show that some seemingly well-defined boundary problems have solutions. However, even if finding such solutions with the computer does not satisfy mathematicians, it will certainly be a major scientific achievement, especially if the solution is in accord with the observational data. I remember how impressed I was several years ago, when the Karman vortices showed up from a numerical integration of the Navier-Stokes equations in a problem of two-dimensional flux against an obstacle. But the computer-aided numerical solution of complex mathematical problems, such as the four-color puzzle, raises a philosophical question and a point of principle. How far are we allowed to help our mind — and to increase our brain's capability — in performing purely mathematical processes and still believe in the validity of the result? On the

other hand, drawing on the wall of the cave or using pencil and paper may be no less ways of increasing the capability threshold of our minds than using computers.

Perhaps the most apparent, even if substantially simpler, achievements of the computers in the recent past, have been their contributions to space exploration and, in particular, solar system exploration. The increasing sophistication of communication links and the almost perfect ground simulation and prediction of actual system performance are direct consequences of two factors: the power of the computer and the fact that the roads of the sky are not only infinite but perfectly smooth and predictable. The mathematical models of the dynamical behavior of the system and of electromagnetic wave propagation are perfectly suited to solutions by computers.

One further remark should be made in relation to the rapid advances in communication theory and computer technology. The deterioration of the general moral environment during the last few decades due to pollution of information is one example of how advanced technology can be misused. Similarly, but on lesser scale, the misuse of computer technology had some negative effects on society at large, in particular, on education, although the extent of the effect is difficult to evaluate.

Finally, in any discussion about computers, I cannot avoid referring, at least briefly, to the question of artificial intelligence.

Actually, I had originally decided not to mention this topic because of two reasons: first, because I do not have sufficient philosophical background, and, second, because friends, who have studied the controversial problem of comparison of the human brain capability with both present and foreseeable computers, suggested that I avoid it carefully. Nevertheless, I am tempted to recall my experiences in the last two months.

In comparing the human brain with an advanced computer, one must compare the common capabilities of the two systems, or at least those capabilities we are now aware of: information storage, access rate to memory, speed of logical data processing, pattern recognition, coordination of activity, elaborate assembling of mechanical systems, learning ability, etc.

In principle, at least, there is no limitation on reproducing in a computer the same capabilities of the human brain. You need only larger and larger computers — and more and more complex instructions. Even emotional sensitivity might be simulated in a computer, as could some sort of creativity, if we consider creativity a random process combined with a judgment capability. According to J.T. Schwartz, the only real deficiency of a computer with respect to the human brain is its present inability to sense useful similarities and to find correlations between different sets of information with sufficient breadth. "If a computer could find worms on a twig with the effectiveness of a bird", writes Schwartz, "it might not fall so far short of the mathematical ability to hunt out interesting theorem." It seems therefore that the difference is only quantitative. The conclusion is astonishing. Humanity is unlimited, constrained only by the infinitesimal pace of evolution. In fact, since all biological systems are basically controlled by chemical and physical processes, it is conceivable that any biological system, no matter how complex, can be reproduced artificially. Thus, if we are able to recreate human intelligence in a machine, we may be able soon to create a superhuman intelligence: present computers with self-reproduction capability, could develop into the earliest specie, "the cristallozoa", of a new form of life, which in turn, could evolve toward a superhuman system.

At this point, not at all convinced of the "gedanken experiment" of Schwartz, and somewhat lost amidst his arguments, I tried to define at least one characteristic of the human brain which could not be simulated on a computer. I have not been very successful! This does not mean that there is none. A mathematician friend gave me a hint: it is possible that some sort of Gödel theorem holds true here, so we will never be able to understand — remaining within our logical system — all the characteristics of the mind. If this is the case, we may never

be able to give a computer instructions we do not know, or to develop capabilities we are not aware of.

More precisely, it appears impossible to build a mathematical model of the "human mind" which has the same relation to the real mind that the Türing automaton has to existing or foreseeable computers. Any attempts to build such an automaton would probably meet the same difficulties we face when trying to show the internal consistency of a system of axioms without a preliminary assumption of the internal consistency of another, larger, axiom system. For any exhaustive description of the human mind, it is almost necessary to have superhuman abilities; and the philosophical implications of such a conclusion are obvious.

5 - Modern mathematics: old and new ideas and concepts for old and new problems

The most promising and innovative developments of mathematics in recent years have taken place in the fields of probability, topology, and combinatorial analysis. What eventually could come out of these new disciplines is unpredictable. My personal feeling is that their potential is very high.

Also significant volume of research has been devoted to logics. However, the results did not bring real advantage to applied mathematical science.

A new logic formulation of the principle of probability theory was badly needed in order to clarify a number of paradoxes. On the other hand, the development of combinatorial analysis and of topology have led to new insights on space and time, the intrinsic geometrical properties of multi-dimensional space, the continuum and countable sets, etc.

Similarly, understanding complex biological molecular structures, may be possible only when new ideas, new concepts, and new imaginative patterns are discovered. The life sciences and behavioral sciences may need just such a renewal of mathematical thought.

Of course, the applicability of mathematics to science is well established on the basis of past experience, especially in physics and chemistry. Consequently, mathematics can be applied to any biological process wich may be considered a physical process such as the dynamic propulsion of single cells or to the many chemical processes which take place in the living cells.

However, when we refer to problems of the life sciences and of human and social behavior, we mean a class of problems of more basic nature — problems related to genetics, operation and evolution of human brain, the origin and evolution of life on earth, etc. In these fields, we need new ideas and new concepts.

Finally, I'd like to make some remarks on two particular fields of modern mathematics. The first concerns what is being called "informatics". Under this name, a large — perhaps too large — number of different activities in the field of applied mathematics have been grouped together. In fact, the tendency to call the same, and sometime old, topics by a new name is clear here; for all sciences must deal with information. The growing overpopulation of researches has led to a differentiation, specialization, and distribution of activities which are more formally than substantially distinct. The result is many people working on the same, simple problems, under different names, while relatively few people work on the more difficult and fundamental problems.

The second field of mathematics is more closely related to my own area of study. I refer to the problem of understanding the evolution and stability of the solar system, including the specific problems of the Kirkwood gaps in the asteroidal belt and the high statistical frequency of exact or close simple commensurability relations among mean motions of planets and satellites. During the last two decades, a large effort has been made within the context of classical analytical dynamics to un-

derstand, at least partially, these problems. The effort has not been very successful. This does not mean that analytical dynamics is wrong. It may only mean that the dynamical model is not sufficiently accurate or, more precisely, that the problem cannot be handled within the model based on Hamiltonian system.

When a process or various processes have been at work for several billion years, as in the case of the evolution of the solar system or the evolution of life on earth, even minor perturbations or minor variations of the physical parameters of the Hamiltonian system may produce major effects on the system. More fundamentally, a qualitative analysis done in the classical model may help, but may never solve a quantitative problem. A clever combination of the traditional analytical approach with a numerical computer-aided approach may be necessary for answering the basic question: Is the Hamiltonian model adequate?

6 - Crisis of science and mathematics

More than 50 years ago, Rutherford made the following remark: "Science is divided into two parts: physics and stamp collecting". Unfortunately, today this remark is still apt, especially now that such a substantial part of modern physics is closer to "stamp collecting" than to true science. In fact, the growing schism between the exact, or mathematical, sciences and the classificatory, or empirical, sciences, as is seen in at least some of the life sciences and physics, has led to an increased, albeit seldom admitted, questioning of our faith in the scientific enterprise.

To give but one example: after the discovery of DNA, the ensuing triumphs of molecular biology were the result of painstaking and enlightened insight into the massive amount of laboratory data. Newtonian mechanics was arrived at only through an elaboration of natural data which required the development and, I should like to say, the invention of concepts not to be found in everyday experience.

What would happen if we tried to develop chemistry from scratch using as our basic ideas only those related to the four Aristotelean elements: earth, air, fire and water? Is it possible that our friends the life scientists, the elementary particle physicists, and even the sometime too imaginative astrophysicists, have unconsciously fallen into a similar trap?

The cosmology introduced by Galileo resulted from a totally original look at, and selection from, experimental data, and Galileo's detractors were partially correct in accusing him of dealing with the imaginary. Yet, the initially imaginary concepts of Galileo in physics and Lavoisier in chemistry, eventually led to today's exact sciences.

Perhaps today's life scientists — and modern physicists, too — need a similar conceptual revolution. Perhaps, in a more general framework, a new type of science and a new type of scientist must develop from this new scientific revolution? Perhaps a new natural philosophy is necessary, one that encompasses those dimensions of the human spirit, such as art and love, which have been considered not pertinent to science since the Renaissance. Indeed, after 500 years of divorce, philosophy and science may again become parts of a single, integrated expression of human spirit. And it could be a new form of mathematics that provides the basic conceptual framework for this second marriage.

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At the University of Padova, where I now teach, Galileo held "the chair in mathematics" and taught practically all the scientific knowledge of his epoch. In fact, his chair was the only chair in the natural sciences at the University of Padova for a century and a half. Only in the middle of the 18th Century was a new chair in "experimental philosophy" added. From a foundation in mathematics, Kepler, Galileo, and Newton began the scientific revolution. The development of mathematics has

followed the development of science, serving as its basic language, its basic instrument, its basic rule of thinking, and, finally, its basic test of understanding.

Any crisis in science leads naturally to the further isolation of mathematicians. They, as well as some scientists in other disciplines, are tempted by pure mathematical research, for it is an activity unlimited by external constraints. While leading to the fulfillment of spiritual needs, typical of any creative activity, mathematics can also produce the gratification of discovering the logical power of our intellect. Who knows? Possibly, this very isolation could contribute to the development of a much-needed new mathematics!

Meanwhile, in its recurrent cycles, nature shows us the deep gaps in our knowledge. And, while we may build solid, and often impressive, bridges across these gaps, they usually bring us to even greater crevasses of ignorance. It is hard to proceed along this difficult trail, passing from euphoria to depression and vice versa. However, we constantly move forward over these obstacles because of the unique prerogative of the human spirit. Call it anxiety or talent, obstinacy or skill, passion or cleverness, it may be the only prerogative which will never be simulated in a computer. It is the same prerogative which prompted the prophet to write: "I prayed and prudence was given to me. I pleaded and the spirit of wisdom came to me. I preferred her to scepter and throne, because all gold, in view of her, is a little sand... Yet, all good things together came to me in her company and countless riches at her hands".

ACKNOWLEDGMENT

Some of the ideas contained in this paper — and perhaps the most interesting ones — do not come from me. I claim only the merit (if any), and take the responsibility of having brought them all together. I learned a great deal from reading the works of F.J. Dyson and J. T. Schwartz, whom, unfortunately, I do not know, and from very fruitful discussions with Gian-Carlo Rota and Ennio DeGiorgi who, fortunately, are my old friends. I am particularly indebted to James C. Cornell of the Smithsonian Astrophysical Observatory for having quickly carried out what looks to me be a very basic editorial revision when I compare the present version with my original one.