SCALING LAWS IN COMPLEX SYSTEMS

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Introduction

There are two types of power scaling distributions in complex systems: a) Discrete distributions, when distribution variable r = 1,2,3... is a natural number (*e.g.*, Zipf's distribution); b) Continuous Distribution, when the distribution variable is continuous (*e.g.*, usual power law). Examples of both distributions are considered below.

1. Zipf's Distribution

The Zipf's distribution related to Riemann zeta function is a discrete distribution that is commonly used in statistical linguistic and has probability density function:

$$P(r,s) = \begin{cases} \dfrac{r^{-s}}{\zeta(s)} & s > 1; \ r = 1,2,3... \\ 0 & Otherwise \end{cases}$$

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$$ Heris is Riemann zeta function.

When it diverges as a sum of harmonic series:

$$\zeta(1) = \sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + ... = \infty$$

Truncated harmonic series or harmonic numbers at large are defined as

$$H_N = \sum_{n=1}^{N} \frac{1}{n} \rightarrow \gamma + \ln(N) = \ln(1.781N)$$

$$where\ Euler\ const:\ \gamma = 0.577,\ e^{\gamma} = 1.781$$

American linguist George Zipf was one of the first who noticed that many data in physical and social sciences can be described by Zipfian distribution.

George Zipf (1902-1950)

Zipf suggested the hypothesis "The Least Effort Principle" to explain the origin of his Laws. Benoit Mandelbrot remarked about Zipf's book, *Human Behaviour and the Principle of Least Effort* "...so many flashes of genius, projected in so many directions..."

1.1. Zipf's First Law

If the words in a text are ordered according to the number of times they appear, then if the position of each word in the list is multiplied by its occurrences, the result achieves to the constant: (word occurrences *P*) x (rank *r*) = *c (*const*)*

$$P(r) \times r = c(const)$$

$$P(r) \approx \frac{P(1)}{r} \quad r = 1, 2, 3 \ldots$$

The first Zipf law in log-log plane represents linear relation

$$\log(occurrences) = \log(c) - \log(rank)$$

Zipf's law for the example, text of the "Declaration of Independence", created by *Mathematica,* is shown in the figure below.
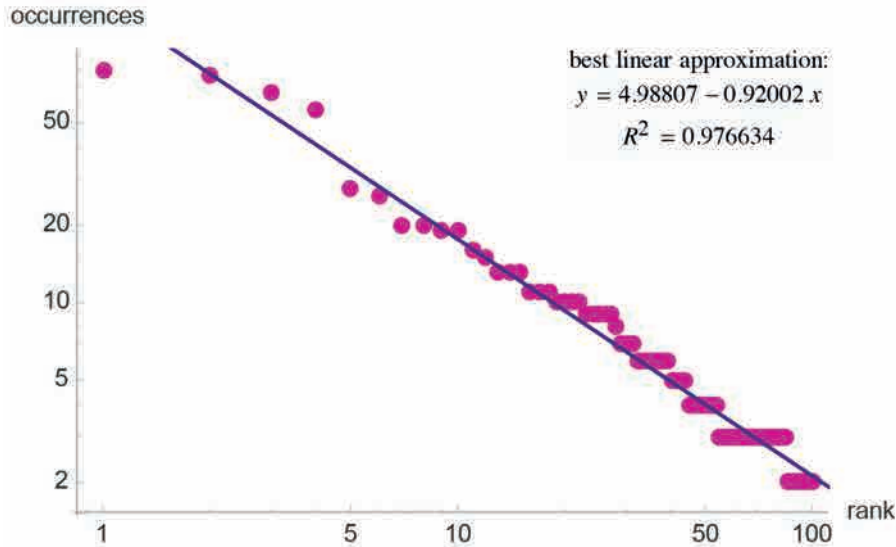

Figure 1. Zipf's law for an example text, adapted from [2]. Omnipresence of Zipf's Law for Complex Systems.

The Least Effort Hypothesis explains the origin of the law as a result of the minimisation of the efforts of speaker and hearer: the tendency of speaker is to use a few words; the tendency of hearer is to demand a specific word (signal) for every event.

Zipf's law today is a well–acknowledged empirical law. Data from various physical and social sciences can be described by Zipf's scaling law. Zipf distribution is ubiquitous in complex systems and well describes different phenomena.

- Frequency of words are Zipf-distributed;
- U.S. firm sizes are Zipf-distributed;
- City sizes are Zipf-distributed;
- Zipf's law governs many features of the Internet;
- Programming language constructs follow Zipf's law;
- Chat robots use Zipf's law.

It is interesting to note that Zipf's book *The Human Behavior and the Principle of Least Effort* and C. Shannon's book *The Mathematical Theory of Communication* were published in 1949 simultaneously. Claude Shannon used Zipf's law to calculate the entropy of an English text.

1.2. Zipf's Second Law

According to Zipf, biological organisms strive to spend as little energy as possible. Zipf argues that in speaking, a considerable amount of energy can be saved if linguistic units that are used very often are kept shorter than those which are used less frequently. Thus, the principle of least effort by means of *economy* controls the structural complexity of linguistic items. Zipf formulated his second law as *the shorter a word is, the more often it will be used*.

2. Continuous Scaling

Any polynomial law $f(x)=ax^b$ where constant a has dimension $a=\dfrac{\dim f}{(\dim x)^b}$ exhibits properties of *scaling* or *scale invariance*. Usually b is called scaling exponent. Function $f(x)$ is shape-invariant with respect to dilatation transformation $x \to \lambda x$

$$f(\lambda x) = a(\lambda x)^b = \lambda^b f(x)$$

According to Euler, $f(x)$ is a homogeneous function of degree b. Differentiating the last equation with respect to λ and $\lambda=1$ we obtain a simple differential equation

$$x f'(x) = b f(x)$$

the solution of which brings us back to the polynomial scaling law. There are tremendously many scaling laws in Nature. A Google search for

"*scaling site:nobelprize.org*"

brings nearly 100 results from the Nobel Foundation.

2.1. Metabolism and Scaling Laws

Metabolic rate is a fundamental quantity in biology. How does the metabolic rate change from mouse to elephant? A first guess is that the metabolic rate would be proportional to some power of mass. If the metabolic rate is directly proportional to the volume (mass) of an animal, then the exponent would be too. If it depends on the surface area, then we could expect dependence. But in 1932 Swiss-American agriculturer Max Kleiber

Max Kleiber (1893-1976)

published a paper [3] with the so–called "mouse to elephant curve" according to which metabolism increases with body mass with expone $b \approx 3/4 = 0.75$, *i.e.* somewhere in between 1 and $2/3 \approx 0.67$, as demonstrated in the figure below.
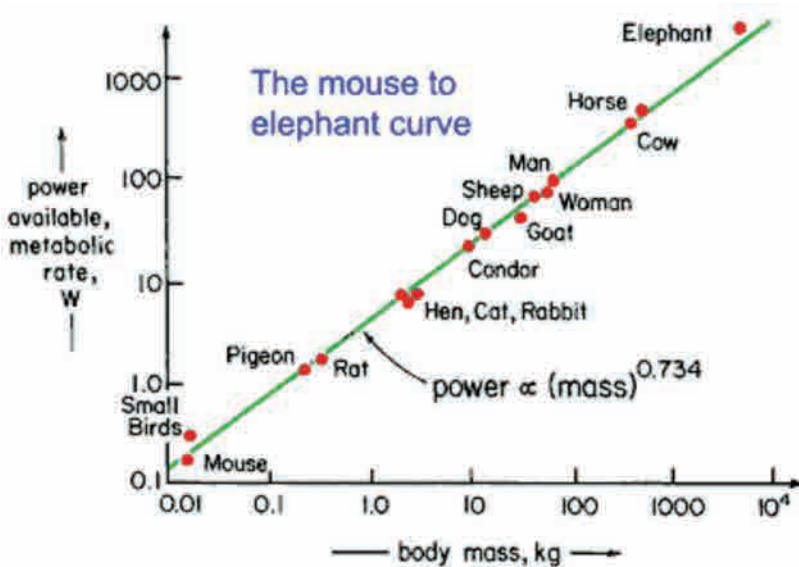

Figure 2. The "mouse-to-elephant" log-log curve shows that the metabolic power of a warm-blooded animal is proportional to its mass to the 0.734 power as established by Max Kleiber in the 1930s. Adapted from K. Schmidt -Nielsen [4].

The dependence of the metabolic rate as as power of the body mass, is known as the Kleiber law. The physical or biological interpretation of 3/4 dependence is not a simple issue. Recent radical ideas of G. West *et. al.* [5]–[7] about *four-dimensional fractal models of life* give impressive support for 3/4 law, but do not satisfy the scepticism of a small minority of supporters of the geometrical 2/3 law. The subject remains in the spotlight of biologists. Recent advancements can probably trigger new insights in the understanding of life's laws.

References

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