Complexity and Analogy in Science
Pontifical Academy of Sciences, *Acta* 22, Vatican City 2014
www.pas.va/content/dam/accademia/pdf/acta22/acta22-heller.pdf

Analogy, Identity, Equivalence

Michael Heller*

Abstract
Category theory, possibly more than any other mathematical theory, has a rich philosophical significance. The reason why it has not been so far exploited by philosophers is that they know it, if at all, only superficially. In the present essay, I shall explore only one aspect of this theory, namely the way it contributes to our understanding of such concepts as: analogy, identity, equivalence. It goes without saying that these concepts play a paramount role not only in many scientific disciplines, but also in philosophy of science and in some fundamental ontological questions. They are notoriously difficult to be defined, and most often are used intuitively or only with the help of purely verbal clarifications. Within the category theory their meaning can be rigorously determined, and their definitions are not arbitrary but imposed, so to speak, by the mathematical context. And even more importantly, these definitions often reveal the variety of meanings never suspected outside the categorical context – the meanings that can doubtlessly be adapted to enrich many traditional philosophical discussions.

1. Introduction
In 1945 Samuel Eilenberg and Sounders Mac Lane published a lengthy paper entitled "General Theory of Natural Equivalences"[9] in which they introduced the concept of category. Many years later Mac Lane confessed that in this paper "they had written what they thought would perhaps be the only necessary research paper on categories" [14, p. 345]. Today every textbook on the category theory quotes this paper as the one that gave birth to one of the most comprehensive mathematical theories.

Category theory, possibly more than any other mathematical theory, has rich philosophical significance. The reason why it has not been so far exploited by philosophers is that they know it, if at all, only superficially. In

*Vatican Observatory, V-00120 Vatican City State; Copernicus Center for Interdisciplinary Studies ul. S-lawkowska 17, 31-016 Cracow, Poland. Correspondence address: ul. Powstańców Warszawy 13/94, 33-110 Tarńow, Poland. E-mail: mheller@wsd.tarnow.pl

the present essay, I shall explore only one aspect of this theory, namely the way it can contribute to our understanding of such concepts as: analogy, identity, equivalence and the like. It goes without saying that these concepts play a paramount role not only in many scientific disciplines, but also in philosophy of science and some fundamental ontological questions. They are notoriously difficult to be defined, and most often are used intuitively or with the help of purely verbal clarifications. One of my motivations to embark on this study was the reading of an essay by R. Brown and R. Porter under the telling title "Category Theory: an Abstract Setting for Analogy and Comparison" [6]. Although my argument goes along slightly different lines, to quote almost literally the abstract of this essay could be the best summary of my aims. "Comparison and analogy are fundamental aspects of knowledge acquisition. One of the reasons for the usefulness and importance of the category theory is that it gives an abstract mathematical setting for analogy and comparison, allowing an analysis of the process of abstracting and relating new concepts. This setting is one of the most important routes for the application of mathematics to scientific problems".

My argument runs as follows. In section 2, I introduce categories and functors in the context of philosophical controversy between objectivist and relativist positions. In section 3, I present natural transformations and adjoint functors. In section 4, I discuss the concept of equivalence, and compare set-theoretic and categorical ontologies. And finally, in section 5, I collect conclusions related to the concept of analogy.

2. Categorical Structuralism

In the philosophy of mathematics there is a long lasting controversy between those who claim that the primary mathematical entities are objects and those who support the view that the primary mathematical entities are structures (roughly speaking, networks of relations) and objects are but "places" within structures (see, for instance [22]). In physics this controversy assumes the form of the question: are space and time collections of object-like points or instances (Newton's view on the absolute space and time), or ordering relations between events (Leibniz's view on the relational nature of space and time) (see, for instance [8])? These polemics are but an echo of what is discussed in metaphysics since Aristotle or even presocratics: are fundamental bricks of reality substances or should the world be regarded as a network of relations? The strength of mathematics consists in its idealization of our intuitive concepts and converts them into formal theories which, in turn, sharpen our understanding of the world. The category theory does this with respect to objectivist versus relational controversies.

A category is a system consisting of:

- Objects: $A, B, C, \ldots$
- Morphisms (also called arrows): $f, g, h, \ldots$ between objects, for instance
$$f : A \to B,$$
A is called domain of f and B is called codomain of f.
- Morphisms can be composed, e.g. if $f : A \to B$ and $g : B \to C$ then
$$g \circ f : A \to B.$$
- The composition is associative, i.e. for all $f : A \to B, g : B \to C, h : C \to D$,
$$h \circ (g \circ f) = (h \circ g) \circ f.$$
- For each object A there is a morphism $1A : A \quad A$, called identity morphism of A, such that
$$f \circ 1_A = f = 1_B \circ f$$
for all $f : A \to B$.

For instance, sets as objects and functions between sets as morpisms form a category, but in general objects need not be sets and morphisms need not be functions.[1]

If in the afore mentioned philosophical discussions we substitute "morphisms" for "relations", we can draw preliminary conclusions important for these discussions. For instance, if we do so with respect to the question: "can objects be entirely eliminated and replaced by a network of relations?", we get a clearly formulated program to be considered in the category theory, namely, can we get rid of objects in the category definition? The result of such an attempt is the so-called objectless category theory [21, pp. 44-46]. The only primitive concept in it is that of morphism, and the axioms assure the composition of morphisms and the existence of the identity morphisms. However, the elimination of morphisms is here only apparent since in fact there is a one to one correspondence between identity morphisms and ob-

[1] For the full definition and more examples see, for instance, [2, chapter 1].

jects, and without identity morphisms categories cannot be defined. But the conclusion that objects are on equal footing with morphisms would also be premature. It turns out that all relevant information on an object can be recovered by considering all morphisms (arrows) ingoing to and outgoing from a given object [15, p. 47]. This recalls Leibniz's idea of monads "which are windowless (we would say they have no internal structure), and the only things that matter are their mutual relationships" [7].

Here we should be warned that in this context the term "relation" is too set-theoretic and too laden with philosophical connotations to correctly render the message coming from the category theory. Traditional relational structuralism is a bottom-up structuralism in which "every relation had to be a relation on some things which, even if they were themselves analyzable into relations, had to be among some other things, ... , and either this process had to stop somewhere (atoms), or an account had to be given of infinite analysis" [3, p. 61]. The categorical structuralism, on the other hand, can be called top-down structuralism. As it is expressed by Awodey, "If we take instead the perfectly autonomous notion of a morphism in a category, we can build structures out of them to our heart's content, without ever having to ask what might be in them" (ibid.). The best solution would be to regard categories themselves as "building blocks" of the categorical structuralism,[2] that is to say to regard them as "objects of a higher order", and look for suitable "morphisms" between them. Such a "morphism" is called functor and is one of the corner stone concepts of the category theory.

A functor from a category $\mathcal{C}$ to a category $\mathcal{D}$ transforms the objects of $\mathcal{C}$ into objects of $\mathcal{D}$, and the morphisms of $\mathcal{C}$ into morphisms of $\mathcal{D}$ in such a way that the structure of the category $\mathcal{C}$ is preserved (for definition see, e.g., [2, pp. 8–9]). The above discussion concerning the relationship between objects and morphisms could in principle be repeated with respect to categories as objects and functors as morphisms. It is obvious that we could proceed further and further along subsequent steps of generalisations, and finally we should consider what is called category of categories. It is a hot topic in the philosophy of mathematics. It was F. William Lawvere who, in his Ph.D thesis [12], tried to consider the category of all categories in the context of the foundations of mathematics, but the very existence of such a category is uncertain. I shall not immerse myself into this discussion (see, for example [17]; this would take us aside from our main line of reasoning.

[2] The discussion on a "categorical structuralism" in the philosophy of mathematics is very much alive; see, for instance [1, 11, 18].

I would like to argue that there are functors that reveal the nature of the categorical structuralism.[3]

3. Natural and Adjoint

In this section we introduce two concepts which play the crucial role in our further analysis; these concepts are: natural transformation and adjoint functor.

Mathematicians often say that a structure or an operation is natural, and usually know what they have in mind, although the concept has not been defined so far. Eilenberg and Mac Lane in their seminal work [9] faced the problem of formally defining the meaning of natural transformation. The result of their effort is this. Let us consider two functors $F, G : \mathbf{C} \to \mathbf{D}$ (of the same variance)[4] from a category $\mathbf{C}$ to a category $\mathbf{D}$. A natural transformation between these two functors, written as $\tau : \mathbf{C} \to \mathbf{D}$, is a family of morphisms in the category $\mathbf{D}$, $\tau_A : FA \to GA$, indexed by objects A of the category $\mathbf{C}$ with some rather obvious conditions supposed to be satisfied (for the definition see, for instance, [23, pp. 90–91]).

Eilenberg and Mac Lane, in their original paper, introduced the concept of an isomorphism of categories more or less in the standard way: A functor $F : \mathbf{C} \to \mathbf{D}$ is said to be an isomorphism if there is a functor $G : \mathbf{D} \to \mathbf{C}$ such that $G \circ F = \mathrm{id}_{\mathbf{C}}$ and $F \to G = \mathrm{id}_{\mathbf{D}}$. A natural transformation between functors F and G is said to be a natural isomorphism if, for each object A of the category C, the morphism $\tau_A : FA \to GA$ is an isomorphism in the category $\mathbf{D}$.

The concept of an adjoint functor was introduced by Daniel Kan in 1958 in [13]. This work truly revolutionised category theory, changing it from a convenient shorthand of some complicated constructions into one of the most fundamental theories of modern mathematics.

Let us consider, as above, two categories $\mathbf{C}$ and $\mathbf{D}$, and let $F : \mathbf{C} \to \mathbf{D}$ and $G : \mathbf{D} \to \mathbf{C}$ be two (covariant) functors as shown below

$$\begin{tikzcd} C \arrow[r, shift left, "\mathrm{F}"] & D \arrow[l, shift left, "\mathrm{G}"] \end{tikzcd}$$

[3] It is a modification of McLarty's "The main point of categorical thinking is to let arrows reveal structure" [16, p. 366].

[4] This means that both functors are either covariant or contravariant.

Let A, B be objects of the category $\mathcal{C}$ and S, T objects of the category $\mathcal{D}$. Let further $\mathrm{Hom}_{\mathcal{C}}(A, B)$ denote all morphisms from the object A to the object B in the category $\mathcal{C}$, and analogously $\mathrm{Hom}_{\mathcal{D}}(S, T)$ in the category $\mathcal{D}$. We want to compare objects A and S, but they live in different categories. Therefore, we either move S, with the help of the functor G, to the category C to compare it with A or, equivalently, we move A, with the help of the functor F, to the category $\mathcal{D}$ to compare it with S. This idea is encoded in the following way

$$\alpha : \mathrm{Hom}_{\mathcal{C}}(A, GS) \to \mathrm{Hom}_{\mathcal{D}}(FA, C)$$

where α is supposed to be a natural isomorphism. If this is the case, F is said to be a left adjoint of G, and G a right adjoint of F, denoted $F \dashv G$ (for the full definition see [23, pp. 148–151] or [2, p. 215]).

As we can see, the functor G is almost an inverse of the functor F, but not quite an inverse [15, p. 159]. Adjoint functors were not known prior to Kan's work, but it has turned out that they play the fundamental role both in the category theory itself and in the whole of mathematics. One often discovers that there exist adjoint functors between categories that are "far away from each other" so that no connection between them was so far suspected. Such a discovery usually leads to new interesting mathematical results.

Suppose we have two structures (categories) between which there exists a pair of adjoint functors. Then one of these structures gives rise to the other structure, and this relationship is reciprocal. Very often, to establish this relationship without the help of adjoint functors would require a long chain of deductions or even would have never been identified.

Suppose we have to solve a problem related to two categories between which there exists a pair of adjoint functors. The merit of formulating the problem in terms of these functors is that if we get a solution, it is guaranteed that the solution is optimal in the strictly defined sense.

4. Set-Theoretic and Categorical Ontologies

In mathematics objects are defined "up to isomorphism", i.e. two isomorphic objects are regarded as two "representations" of the same object. In other words, it is isomorphism that gives to an object its identity. We could call this "set-theoretic ontology".[5] It was Alexander Grothendieck who noticed that this ontology is not suitable for categories: there are some categories that are

[5] The term "ontology" is used here in the sense close to that proposed by W. Quine [19].

not isomorphic with each other that should nevertheless be regarded as "the same" from the categorical point of view[10]. He thus proposed the following definition. Two categories $\mathcal{C}$ and $\mathcal{D}$ are equivalent if there are functors $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$ such that $id_{\mathcal{C}} \cong G \circ F$ and $id_{d} \cong F \circ G$ are natural isomorphisms. The fact that, in this definition, the concept of natural isomorphism appears, makes it natural for the category theory. If we use the equivalence in the Grothendieck sense to equipp categories with their identity, we can speak about a categorical ontology.

It is easy to see that two equivalent (in the Grothendieck sense) functors F and G are adjoint, $F \dashv G$ (but not necessarily *vice versa*). As the consequence, the equivalence of categories is a special case of adjointness. Therefore, if two categories C and D are equivalent, there exists a pair of adjoint functors $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$, but these functors need not be the inverses of each other, i.e. they need not define an isomorphism of categories. If they do, the functors are trivially equivalent. Here we have a surprise: it can happen that this deviation from triviality can lead to interesting theorems.[6] We can see here that the "space of categories" is not a loose agglomeration of categories, but rather a highly structured "field" that reacts on "perturbations" of its substructures in a sophisticated manner.

We can go even further in comparing the "set-theoretic world" with the "categorical world". In the set-theoretic approach, the *univers de discourse* of mathematics is the "space of all sets" (we abstract here from paradoxes of the set-of-all-sets type); in the categorical approach the *univers de discourse* of mathematics is the "space of all categories" as mentioned above (we abstract here from the discussions around the category of categories). Therefore, a single category is a point in this space. But the category of all sets as objects and all functions between them is only a one category among many others, that is to say, just one point in the space of categories. This shows vastly different perspectives of both these approaches.

Could this vast multiplicity of categories be made of an immense number of families of sets and functions between them? If so, the categorical ontology would finally be reducible to the set-theoretic ontology. This problem has been, at least in part, elucidated by some consequences following

[6] Here is an example: Let C be a category the objects of which are locally compact Abelian groups and the morphisms are continuous group homomorphisms. Let also *Cop* be the category opposite to C. These two categories are equivalent (in the Grothendieck sense), and this equivalence is nothing else but the well known Pontryagin theorem. For more examples see [20].

from the so-called Yoneda lemma. The following remarks should give the reader the general idea (for details see [2, pp. 185-192]).

Let $\mathcal{C}$ be any locally small category.[7] It can be represented by functors from the category $\mathcal{C}^{op}$ opposite[8] to $\mathcal{C}$ to the category **Sets** of sets as objects and functions as morphisms. More technically, the representation is given by a functor, called Yoneda embedding,

$$y : \mathcal{C} \to \mathbf{Sets}^{\mathcal{C}^{op}} .$$

It says that the category C can be identified with a subcategory of functors from $\mathcal{C}^{op}$ to **Sets**. In other words, any (locally small) category can be represented in terms of a functor category. Here is a short comment given by Marquis: "This might seem innocuous but it constitutes an extremely important shift that has tremendous implications, both mathematically, and philosophically, that is in the way one thinks about what mathematics is *about*. For now, objects of a category are not fundamentally structured sets, they are first and foremost *functors*" [15, p. 105]. Categorical ontology is a functorial ontology. Even if the majority (at least all locally small categories) can indeed be translated into families of sets and families of functions between them, the categorical perspective is totality different: the space of categories is "spanned" by functors rather than by functions, and this makes the difference. Functors are not only much richer, but also much more flexible. A lot of possibilities that are excluded in the set-theoretic ontology in the categorical ontology are quite "ordinary". Many of our scientific and philosophical concepts if looked upon from this new perspective could reveal their unexpected aspects. This is also true as far as the concept of analogy is concerned.

5. Analogies

The concept of analogy belongs to the family of those concepts that being important for scientific or philosophical discourse, are at the same time fuzzy and change their meaning depending on the context. In spite of some heroic attempts, undertaken mainly by philosophers, linguists and

[7] A category C is locally small if, for all its objects X, Y the collection of all morphisms from X to Y is a set.

[8] The category *Cop* , opposite to the category C, has the same objects as C, and the same arrows but in the reversed direction.

logicians,[9] we use them by basing them on our intuition rather than on some hard analyses. There is a strong connection between our intuitions and the set-theoretic ontology. No wonder since a set-theoretical thinking was created by formalizing and idealizing our every-day intuitions. My view is that our intuitions, contaminated by a set-theoretic thinking, do not grasp the full content of the concept of analogy as it reveals itself in the manifold of its applications. To this end the category theory seems to be much better suited. The concept of analogy seems to be, from its very nature, multi faced and adapting its meaning to various situations. It is a dynamic concept. This is why the categorical ontology seems to offer more effective means to deal with the issue of analogy.

Looking at analogy in the categorical environment and with the help of categorical tools is so promising that it would require a more profound study; here I offer only a few hints or remarks suggesting some perspectives:

- Classically analogy is defined or described in terms of relations between objects; categorical concept of morphism, together with its "dom inance" over objects, enables us to disclose more shades of the notion of analogy. For instance, the very concept of object in the categorical setting reveals its dynamical nature: the usual question "what is object?" is replaced by the question "what is object doing?".

- Even more so as far as the concept of functor is concerned. Owing to this concept we can speak about "analogous" categories, and the category concept is so rich that it embraces structured sets, relations, relations between relations, abstract and concrete processes, etc., etc. One can deal with this variety of situations not only at the intuitive level, but also in a mathematically precise way. If we want to decide whether two things or situations deserve the name "analogous", we must compare them. In the category theory we compare categories with the help of functors, and the enormous variety of functors offers a great richness of comparing methods. Since there are functors between functorial categories (i.e. such categories the objects of which are functors), we can also explore "analogy between analogies". It seems that natural transformations and adjoint functors are especially suitable to this end.

[9] From the formal point of view analogy was analysed by the Polish School of Logic (see [24]). The most known are the works of J.M. Bocheński [4, 5].

- A natural transformation compares two functors between two categories. The way of comparing seems complicated, but it effectively detects a certain *affinity* between them. It is tempting to call this affinity of functors, at least in some situations, analogy.

- If we agree to relate analogy to natural transformations, we must acknowledge that analogy is involved, via the natural isomorphism α (section 3), into the concept of adjointness. The fact that the isomorphism α is natural means that if we transform the object A into an object B (in the category $\mathcal{C}$), or the object S into an object T (in the category $\mathcal{D}$), the correspondence (analogy) between the two hom-sets will be preserved. Two adjoint functors between two categories are "almost inverse of each other, but not quite an inverse". If they were inverse, the two categories would be isomorphic, and the fact that they are not, allows us to call them analogous. As put by Brown and Porter, "the partial matching, via a comparison, of the properties of A and B leads to analogy" [6, p.3].

I am far from thinking that with natural transformations and adjoint functors the analogy problem in relation to category theory has been closed; they are only examples of what could be achieved.

One more lesson from the above analysis. After all, it is not that important how we call our concepts (analogous or not) as long as we have effective tools to compare them, and it is category theory that offers such tools. We should only learn how to use categorical models in the service of philosophical investigations. Mathematical tools are much richer than our every-day intuitions and purely verbal distinctions; they are able to reveal unexpected aspects of reality.

References

[1] Awodey, S., "Structure in Mathematics and Logic. A Categorical Perspective", *Philosophia Mathematica* 4(3), 1996, 209-237.

[2] Awodey, S., *Category Theory*, second edition, Oxford University Press, Oxford-New York, 2010.

[3] Awodey, S., "An Answer to Hellman's Question: 'Does Category Theory Provide a Framework for Mathematical Structuralism'", *Philosophia Mathematica* 12(3), 2004, 54-64.

[4] Bocheński, J.M., "On Analogy". *The Thomist* 11, 1948, 424-447; reprinted and corrected in: *Logico-Philosophical Studies*, ed. E. Menne, D. Raidel, Dordrecht, 1959.

[5] Bocheński, J.M., *The Logic of Religion*, New York University Press, New York, 1965.

[6] Brown, R., Porter, T., "Category Theory: an Abstract Setting for Analogy and Comparison", www.math.bangor.ac.uk/research/ftp/cathom/0.510.pdf

[7] Cartier, P., "A Mad Day's Work: from Grothendieck to Connes and Kontsevich. The Evolution of Concepts of Space and Symmetry", *Bulletin of the American Mathematical Society (New Series)* 38(4), 2001, 389-408.

[8] J. Earman, *World Enough and Spacetime*, MIT Press, Cambridge, Mass. London, 1989.

[9] Eilenberg, S., MacLane, S., "General Theory of Natural Equivalences", *Transactions of the American Mathematical Society* 58, 1945, 231-294.

[10] Grothendieck, A., "Sur quelques points d'algèbre homologique", *Tôhoku Math. J.*, 9 (2), 1957, 119-221.

[11] Hellman, G., "Does Category Theory Provide a Framework for Mathematical Structuralism", *Philosophia Mathematica* 11(3), 2003, 129-157.

[12] Lawvere, F.M., "The Category of Categories as a Foundation for Mathematics", *Proceedings of the Conference on Categorical Algebra*, La Jolla 1965, Springer-Verlag, New York, 1966.

[13] Kan, D.L., "Adjoint Functors". *Transactions of the American Mathematical Society* 87, 1958, 294-329.

[14] MacLane, S., "Concepts and Categories in Perspective", in: *A Century of Mathematics in America*, American Mathematical Society, 1988, pp. 323-365.

[15] Marquis, J-P., *From a Geometrical Point of View*, Springer, 2010.

[16] McLarty, C., "The Uses and Abuses of the History of Topos Theory", *Brit. J. Phil. Sci.* 41, 1990, 351-375.

[17] MacLarty, C., "Axiomatizing a Category of Categories", *Journal of Symbolic Logic* 56, 1991, 1243-1260.

[18] McLarty, C., "Exploring Categorical Structuralism", *Philosophia Mathematica* 12(3), 2004, 37-53.

[19] Quine, W., "On What There Is", in: *From a Logical Point of View*, Harvard University Press, Cambridge, Mass., 1964, pp. 1-19.

[20] Resende, P., "Quantales as Geometric Objects: Symmetry beyond Groupoids", arXiv:math/0506451.

[21] Semadeni, Z., Wiweger, A., *Wstęp do teorii kategorii i funktorów* (Introduction to Category Theory and Functors), PWN, Warszawa, 1978 (in Polish).

[22] Shapiro, S., *Thinking about Mathematics*, Oxford University Press, Oxford, 2000.

[23] Simmons, H., *An Introduction to Category Theory*, Cambridge University Press, Cambridge, 2011, pp. 90-91.

[24] Wolak, Z., *Koncepcje analogii w Kole Krakowskim* (The Concepts of Analogy in the Cracow Circle), Biblos, Tarnów, 2005 (in Polish).