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DISCOVERING THE WORLD STRUCTURE AS A GOAL OF PHYSICS

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SUMMARY

The structuralist view on mathematics, claiming that mathematics is ‘about structures and their morphology’, is well founded in both mathematical practice and metatheoretical investigation, although it is difficult to formulate in a rigorous manner. Assuming the structuralist view on mathematics, we look for its consequences for interpreting physical theories.

Empirical successes of physical theories support their realistic interpretation, but discontinuities in the world image, often caused by replacing a successful physical theory by another even more successful theory, support their anti-realistic interpretation. However, if we remember that the mathematical structure of the old theory can be obtained, as a limiting case, from the mathematical structure of the new theory, we recover a continuity at the level of structures (even if there is no continuity at the level of respective world images). This supports the view that physical theories are ‘about the structure of the world’ or, more precisely, that the method of physics grasps only the world structure. From the philosophical point of view, two possibilities can be envisaged: either (1) structures discovered by physical theories are structures of something, i.e., there is a ‘structured stuff’, but this stuff is transparent for the physical method (epistemic structuralism), or (2) such a stuff does not exist (ontic structuralism). To decide between these two possibilities one should go beyond the analysis of the physical method and appeal to metaphysical or logical reasons.

In the case, when one physical theory admits more than one mathematical formulation (which is not infrequent situation in physics), we should assume that these formulations are but various representations of an abstract structure, and that it is this abstract structure that reflects or approximates the Structure of the World.

1. Introduction

The line of reasoning adopted in this paper is the following:

– Mathematics is a science of structures, i.e., the world discovered (or constructed?) by mathematicians consists of structures and relations between structures.

– Physics employs mathematical structures to model the world.

– Therefore, the world as discovered (or constructed?) by physicists consists of mathematical structures interpreted as structures of the world.

As we can see, the above strategy is based on methodology: by analysing the method of both mathematics and physics we try to gain some important information about the world. And the information is that the world, as discovered (or constructed) by physics, is purely structural. Only its structural aspects are visible for the mathematical-experimental method of physics. A material stuff supporting relations, that constitute the structure, if it exists, is transparent for this method.

This is not a new philosophy of physics. It is known for a long time that there are interactions between physical objects rather than the objects themselves that are accessible for the method of physics. And the word 'interactions' is just another name for 'relations constituting the structure'. However, in the last decades, this philosophy of physics was subject to a detailed scrutiny, and became a subject-matter of a long dispute, out of which new and illuminating arguments have emerged.

The above presented 'line of reasoning' determines the organization of my material. First, I focus on the 'mathematical structuralism', then I discuss its use in physics to finally draw some conclusions concerning what could be called the structuralist ontology of the world.

In pursuing this plan, I base my presentation on the above-mentioned debate which is still going on in the foundations of mathematics and in philosophy of physics.

2. Structuralist Approach to Mathematics

It would be difficult to find a mathematician who, for the first time, used the word 'structure' to denote the subject-matter of mathematical inquiry. Nowadays, the saying that mathematics investigates structure is a kind of common wisdom among mathematicians. Any deeper study of mathematics, especially of geometry, creates a persistent impression of an architectonic totality, the elements of which acquire their meaning only through their relationship with other elements and the underlying plan. No wonder that structuralist ideas had appeared long before they became a part of an official current in the philosophy of mathematics. The concept of structure (understood in a more technical way) invaded first the abstract algebra, and from algebra it proliferated – mainly owing to the works of Nicolai Bourbaki – to almost the whole of mathematics. Some structuralist view could be traced back to the writings of Hilbert, Bernays and Quine, but it is often Michael Resnik who is credited with initiating the 'structuralist movement'. In his paper 'Mathematics as Science of Patterns: Ontology and Reference',[1] considered by many as a kind of structuralist manifesto, Resnik declared that in mathematics we are never dealing with objects equipped with 'inner properties', but only with structures. The objects, studied in mathematics, are but 'structureless points' or positions within structures. Besides their place in a structure, they are devoid of any individuality and any property independent of the structure. He wrote:

> The objects of mathematics, that is, the entities which our mathematical constants and quantifiers denote, are structureless points or positions in structures. As positions in structures, they have no identity outside of a structure.[2]

The thesis, understood intuitively, that mathematics is a science of structure, does not excite any emotions, but if one wants to make it more precise, opinions start to diverge, and technical discussions replace a consensus. As far as these technical aspects of the problem are concerned two mathematical theories seem to be especially relevant: set theory and category theory.

2.1. *Mathematical Structuralism and Set Theory*

The great majority of current works in mathematics is based on the assumption that mathematics is founded on set theory. No wonder, therefore, that when we think on the structural approach to mathematics, the first question that should be answered is whether the set theoretic definition of number (which 'evidently' are objects!) could be rephrased in a structuralist manner. Paul Benacerraf, in his paper 'What Numbers Could

[1] *Nous*, 15 (1981), pp. 529-550.
[2] *Ibid.*, p. 530.

Not Be?',[3] addresses this question. He points out that, for instance, number '3' can be identified with different set theoretic objects (e.g. with {{{0}}} in agreement with Zermelo's approach or with {0, {0}, {0}} in agreement with von Neumann's approach), and any such 'object' can serve as a representative of number '3' provided it satisfies certain structural relations. The mathematician's interest does not go beyond structural properties. It is impossible to identify an 'object' independently of the role it plays in the structure. Referring to this view, Charles Parsons[4] speaks about the eliminative structuralism: all utterances on objects should be understood as utterances on structures, and one should look for mathematical procedures that would allow one to eliminate from the discourse all mention on objects by substituting for them structural terms.

Structuralist approach to the philosophy of mathematics is often identified with the Platonist approach. In fact, Resnik, in his seminal paper, claimed that it is the structuralist approach that puts the Platonist philosophy on the firm ground. In his view, the belief in objects is incompatible with the Platonist philosophy since 'no mathematical theory can do more than determine its objects up to isomorphism', and

> [t]he Platonist seems to be in the paradoxical position of claiming that a given mathematical theory is about certain things and yet [he is] unable to make any definitive statement of what these things are.[5]

However, this link between structuralism and Platonism was questioned by Steward Shapiro.[6] He argues that since 'mathematics applies to reality through the discovery of mathematical structures underlying the non-mathematical universe', the latter is ontologically prior with respect to mathematical structures. Although Shapiro's views are open to a criticism,[7] it is important to realize that mathematical structuralism is not to be identified with mathematical Platonism.

The strict definition of structure can be found in abstract algebra. Here structure is understood as a domain, possibly with some distinguished elements, on which some relations and functions are defined and

[3] In P. Benacerraf, H. Putnam (eds.), *Philosophy of Mathematics: Selected Readings*, 2nd edition (Cambridge, Cambridge University Press, 1983), pp. 272-294 (1st edition, 1965).

[4] 'The Structuralist View of Mathematical Objects', *Synthese*, 84 (1990), pp. 303-306.

[5] *Op. cit.*, p. 529.

[6] 'Mathematics and Reality', *Philosophy of Science*, 50 (1983), pp. 523-548.

[7] See, e.g., Ch. Chihara, *Constructability and Mathematical Existence* (Oxford, Clarendon Press, 1990).

satisfy suitable axioms. Examples of such structures are: group, vector space, module, linear algebra. However, if one wants to use this structure definition for philosophical purposes, one immediately meets the following difficulty: the domain entering the structure definition is a set, and relations and functions on this domain are defined in terms of Cartesian products of sets. It is, therefore, evident that this structure definition presupposes the existence of sets, and cannot serve as a tool to replace the 'set philosophy' with a structural approach. Problems of this sort make people look for a solution in category theory.

2.2. *Mathematical Structuralism and Category Theory*

One of the main motives to create category theory was to grasp the essence of structure, S. Eilenberg and S. MacLane[8] aimed at this topic not for philosophical reasons but to elaborate a tool to investigate dependences between various mathematical theories.

For precise definitions introducing the concept of category the reader should look in the suitable mathematical literature,[9] here we give only an 'intuitive description', sufficient to follow further analyses. By a *category* one should understand:

(1) a class of objects: U, V, W, Z,...

(2) a class of *morphisms*: *Mor (U, V)* for every pair (U, V) of objects,

(3) a composition of morphisms.

Morphisms could be viewed as mappings between objects, although neither objects have to be sets, nor morphisms have to be mappings between sets. Of course, objects, morphisms and the operation of composition of morphisms must satisfy suitable axioms.

Common examples of categories are: category *Ens* with any sets as objects, and any mapping between them as morphisms; category *Top* with topological spaces as objects and continuous mappings as morphisms; category *Gr* with (non-empty) group as objects, and group homomorphisms as morphisms.

An important role in category theory is played by *functors*; they could be thought of as mappings between categories that preserve some of their

[8] 'General Theory of Natural Equivalences', *Trans. Amer. Math. Soc.*, 58 (1945), pp. 231-294.

[9] A good introductory course could be found in R. Geroch, *Mathematical Physics* (Chicago-London, The University of Chicago Press, 1985).

properties. For instance, the *forgetting functor* goes from *Gr* to *Ens* in such a way that it ascribes to every group the set on which the group is defined (this functor preserves the set structure but forgets the group structure).

In category theory the stress is laid on morphisms and functors rather than on objects. Motivated by this intuitive idea, suggestions were made that category theory could provide new foundations for mathematics.[10] It is interesting to notice that objections against this view were directed not against structuralism, but rather in defense of the thesis that it is only set theory that could provide necessary tools to reach this goal.[11] However, this claim seems to be unjustified because of the existence of the topos category. It is a category that unifies in itself the notion of a 'generalized set' with the notion of a 'generalized space'.[12] There are many topoi which can be used for various purposes. For example, one can define a topos that does not require the law of the excluded middle, or a topos without the axiom of choice. Among these topoi there is one which is equivalent to the standard set theory. Because of this elasticity the theory of topoi combines in itself properties of geometry and logic. It is therefore evident that everything that can be achieved for the foundations of mathematics with set theory can also be done with category theory. Could category theory do more than set theory in creating structuralist foundations of mathematics? The situation in this respect is not clear.

The evolution of MacLane's views in this respect is significant. In his earlier work[13] he expressed a more radical view, and hoped that category theory would be able to provide strict and better foundations of mathematics than those based on set theory. Later on, when these hopes had not materialized, he had to change his standpoint. Out of the failure he made a philosophy. Mathematics has a 'Protean character', in the sense that it explains why it can have no foundations. Although category theory cannot provide strict foundations for mathematics, it has a 'foundational significance' since it organizes the whole of mathematics so that we are able to speak about mathematical structures and structures of these structures.[14]

[10] S. MacLane, *Mathematics: Form and Function* (New York, Springer, 1986).

[11] J. Mayberry, 'What is Required of a Foundations of Mathematics?', *Philosophia Mathematica*, 2 (1992), pp. 16-35.

[12] For precise definitions see, for instance: S. MacLane, I. Moerdijk, *Sheaves in Geometry and Logic: A First Introduction to Topos Theory* (New York, Springer, 1992).

[13] *Mathematics: Form and Function*, *op. cit*.

[14] S. MacLane, 'The Protean Character of Mathematics', *The Space of Mathematics*: J. Echeverra, A. Ibarra, J. Mormann, De Gruyter (eds.), (New York, 1997), pp. 3-12.

In this sense, 'mathematics has as its subject-matter structures and their morphology'.[15] To express these ideas in a more strict language people start to speak about the 'category of all categories'. Attempts to give its precise definition are involved in many technicalities, the discussion of which would lead us beyond the scope of the present paper.

Finally, let us mention that there were attempts at creating a formal theory of structure. For instance, Shapiro proposed a suitable axiomatic system.[16] However, since his system imitates that of a set theory, it is involved essentially in the same interpretative difficulties.[17]

2.3. *Tentative Conclusions*

Let us collect some tentative conclusions following from the above analysis. The structuralist interpretation of mathematics is well-founded in both everyday mathematical practice and in metatheoretical investigations. However, it is notoriously difficult to express it in a strict but adequate way. Moreover, the border-line between the 'objectivist' and 'structuralist' views is rather fuzzy, and if one wanted, in some cases, to determine this border more precisely, one would have to look at what a given mathematician is doing rather than to listen to his or her declarations.

We should distinguish the weaker and stronger versions of mathematical structuralism. Adherents of the weaker version do not negate the existence of mathematical objects but treat them as 'places' in a structure. These places are either devoid of an 'inner structure', or their structures are entirely determined by the structure in which they are substructures. Adherents of the stronger version claim that either the concept of structure does not require the existence of objects at all (ontological structuralism), or that if objects do exist, they are not cognizable. It is clear that the latter distinction have strong metaphysical bearing, and goes beyond the foundations of mathematics.

In the following, we shall work with the assumption that mathematics is 'about structures and their morphology', and look for the consequences

[15] E. Landry, 'Category Theory: The Language of Mathematics', scistud.umkc.edn/psa 98/papers/.

[16] See, S. Shapiro, *Philosophy of Mathematics. Structure and Ontology* (New York-Oxford, Oxford University Press, 1997).

[17] For a detailed analysis see: K. Wójtowicz, *Spór o istnienie w Matematyce (Dispute on the Existence in Mathematics)*, in Polish (Warszawa, Semper, 2003), chapter 10.C.

of this assumption as far as the nature of physics is concerned. We shall, in principle, understand this assumption in the weaker sense, but we shall also occasionally try to see the implications of its stronger forms.

3. The Best of Both Worlds

If mathematics is a science of structures, and physics' main tool in investigating the world is by constructing its mathematical models, the question arises in which sense the structuralist view on mathematics transfers into our knowledge of the physical world. It is a happy circumstance that in order to elucidate this question, we can make use of a recent discussion in the philosophy of physics concerning these matters. Although this discussion was focused on a link between the structuralist approach to physics and its realist or antirealist interpretations, it also significantly contributed to a better understanding of the structuralist standpoint itself.

Remarks made by Jeremy Butterfield, a philosopher of science, and Chris Isham, a theoretical physicist, in their recent paper on quantum gravity[18] can serve as a good introduction to this discussion. When warning against dangers of a far-fetched realism in interpreting physics, they claim that every major innovation in physics changes its very subject-matter. In their opinion, this follows from the fact that physics aims at giving a complete description of the world. Both the common-sense knowledge and various scientific disciplines, such as chemistry or history, are subject to major changes, however,

> ... no enthusiast of such a science or discipline is mad enough, or imperialist enough, to believe that it gives a complete description of its subject-matter. There are always other conjuncts to be had, typically from other disciplines.
>
> Not so, we submit, for physics, or at least theoretical physics: whether it by madness, imperialism, or part of what we mean by 'physics', physics *does* aspire to give just this sort of complete description of its subject-matter. And this implies that when 'other

[18] J. Butterfield and C.J. Isham, 'Spacetime and the Philosophical Challenge of Quantum Gravity', *Physics Meets Philosophy at the Planck Scale*, C. Callender, N. Huggett (eds.), (Cambridge, Cambridge University Press, 2001).

> conjuncts arrive' – i.e., new facts come to light additional to those given by the most complete available physical description – it is more reasonable to construe the new facts as involving a change of subject-matter, rather than as an additional piece of doctrine about the old subject-matter.[19]

To put this short: the subject-matter of physics presupposes a certain ontology of the world. If the subject-matter changes, the assumed ontology changes as well. And this implies that we should 'beware scientific realism'.[20]

The recent discussion in philosophy of physics, exactly on this problem, was initiated by John Worrall's paper entitled 'Structural Realism: The Best of Both Worlds?'.[21] He starts his analysis with the well-known argument on behalf of scientific realism, which he calls 'no miracles' argument:

> Very roughly, this argument goes as follows. It would be a miracle, a coincidence on a near cosmic scale, if a theory made as many correct empirical predictions as, say, the general theory of relativity or the photon theory of light *without* what that theory says about the fundamental structure of the universe being correct or 'essentially' or 'basically' correct.[22]

We thus have two conflicting views: realism and anti-realism, and both of them look for support in the progress of physics: accumulation of its results supports realism, and discontinuities in its description of the world supports anti-realism. Worrall asks:

> Is it possible to have the best of both worlds, to account (no matter how tentatively) for the empirical success of theoretical science without running foul of the historical facts about theory-change?[23]

We should distinguish the continuity or discontinuity in the evolution of physics at the level of empirical results and at the level of world's description. There are no major problems, as far as the 'essential cumulativity' at the empirical level is concerned; it is a 'non-cumulative kind at the top theoretical levels' that creates serious problems. One of the answers to these problems could be an instrumentalist view of science, according to which theories make 'no real claims beyond their directly empirical conse-

[19] *Ibid.*
[20] This is the title of a subsection in the above quoted paper by Butterfield and Isham.
[21] *Dialectica*, 43 (1989), pp. 97-124.
[22] *Ibid.*, p. 101.
[23] *Ibid.*, p. 111.

quences', but this is exactly what Worrall wants to avoid. The solution he proposes is called (by him) *structural realism*. He writes:

> The role in the history of physics seems to be that, whenever a theory replaces a predecessor, which has however itself enjoyed genuine predictive success, the 'correspondence principle' applies. This requires the *mathematical equations* of the old theory to re-emerge as limiting cases of the mathematical equations of the new.[24]

Even if there is a discontinuity in the 'world images' given by the old and new theories, there is a substantial continuity in their respective mathematical structures. It 'is not evidence for full-blown realism – but instead only for structural realism'.[25]

The change from Fresnel's wave theory to Maxwell's electromagnetic theory is an often discussed example in this context.[26] There is certainly a drastic 'ontological change' between light as a periodic disturbance in an elastic medium (Fresnel) and light as an excitation of the 'disembodied' field (Maxwell). However, the continuity is recovered as soon as we look at this 'theory shift' from the structuralist point of view.

> Fresnel's equations are taken over completely intact into the superseding theory – reappearing there newly interpreted but, as mathematical equations entirely unchanged.[27]

The latter situation is rather exceptional in the history of physics. A more common one, and entirely sufficient for the structuralist approach, is that the equations of the old theory appear as limiting cases of the new theory. Since mathematical equations describe structures, the example of Fresnel and Maxwell theories exhibits

> cumulative growth at the structural level combined with the radical replacement of the previous ontological ideas. It speaks, then, in favour of a *structural* realism.[28]

[24] *Ibid.*, p. 120.

[25] *Ibid.*, p. 121.

[26] It was already Poincaré who quoted this example (*Science and Hypothesis*, Dover, 1905, pp. 160-162) 'to argue for a general sort of *syntactic* or *structural realism* quite different from the anti-realist instrumentalism which is often attributed to him' (Worrall, op. cit., p. 117).

[27] Worrall, *op. cit.*, p. 120.

[28] *Ibid.*

4. What Is Structural Realism?

In his response to Warrall's paper, James Ladyman[29] quotes the structuralist views of Bertrand Russell and Grover Maxwell who claimed that

> the objective world is composed of unobservable objects between which certain properties and relations obtain; but we can only *know* the properties and relations of these properties and relations, that is the *structure* of the objective world.[30]

This view is qualified as an *epistemological* (or *epistemic*) *structuralism*. Ladyman himself opts for something more. He argues that

> structural realism gains no advantage over traditional scientific realism if it is understood as merely an epistemological refinement of it, and that instead it ought to be developed as a metaphysical position.[31]

Ladyman thinks that to cope with this problem one must change from the syntactic, or 'received', view to the semantic view on scientific theories. According to the syntactic view, scientific theory is but a system of propositions, in the ideal case, an axiomatic system suitably interpreted, whereas according to semantic approach

> theories are to be thought of as presenting structures or models that may be used to represent systems rather than partially-interpreted axiomatic systems. Theories are not collections of propositions or statements, but are 'extra-linguistic entities which may be described or characterised by a number of different linguistic formulations'.[32]

The distinction between epistemic structuralism and ontic structuralism was made sharp by Michael Esfeld who writes:

> If physics tells us only about the way in which things at the basic level of the world are related to each other, two different metaphysical positions are open: (1) The things at the basic level have intrinsic properties of which we cannot gain any knowledge insofar as

[29] 'What Is Structural Realism?', *Studies in the History and Philosophy of Science* 29, 1998, pp. 409-424.

[30] *Ibid.*, p. 412.

[31] *Ibid.*, p. 411.

[32] *Ibid.*, p. 416. The last sentence is a quotation from: F. Suppe, *The Structure of Scientific Theories* (Chicago, University of Illinois Press, 1974), p. 221.

they are intrinsic. (2) The relations in which they stand are all there is to the things at the basic level.[33]

View (1) is called the *epistemic* (or *epistemological*) structuralism; view (2) is called the *ontic* (or *ontological*) structuralism.

In this paper, I am not so much concerned about the realism – anti-realism dispute, but rather about structuralist interpretation of physics. And from this point of view, the following conclusion should be regarded as well-founded: From the analysis of the mathematical-empirical method of physics it follows that physical theories are about structures of the world. Two interpretations of this fact are possible: either structures, discovered by physical theories, are structures of something, i.e., there is a structured stuff, but this stuff is transparent for the method of physics (epistemic version), or such a stuff is absent (ontic version). When adopting the second version, we could speak about objects between which structural relations obtain, but such objects should be understood as 'empty places' in a structure (empty – because devoid of any 'intrinsic properties'). To decide between these two versions, we should go beyond the analysis of the physical method, and appeal to metaphysical or logical reasons (for instance, to the claim that there cannot be structural relations without the relata), or to invoke some 'strategic principles' (like Ockham's rasor to claim that 'undetectable stuff' is not necessary).

5. Representation Invariants

An unavoidable question concerning structuralism is what should we understand by structure. An easy answer would be that the structure presupposed by a given physical theory should be identified with (or somehow correlated to) the mathematical structure this theory employs. Unfortunately, this answer meets the following difficulty. In physics, there are theories that admit more than one, apparently different, mathematical formulations. For instance, classical mechanics can be expressed in terms of the 'action at a distance', in terms of variational principles, in terms of space-time curvature or with the help of field theoretical methods. Quantum mechanics also admits several representations: in terms of operators on a Hilbert space, in

[33] M. Esfeld, 'Do Relations Require Underlying Intrinsic Properties? A Physical Argument for a Metaphysics of Relations', *Metaphysica*, 4 (2003), pp. 5-26.

terms of C^*-algebras, in terms of Feynman path-integrals, or in terms of density matrices. This creates the real problem for the traditional realism, but can be a chance for the structuralist interpretation; namely, we can claim that, in a case of many mathematical formulations of the same physical theory, we have various representations of *the same structure*. This was the view of Dirac and Weyl with regard to two different 'pictures' of quantum mechanics – the wave picture created by Schrödinger and the matrix picture created by Heisenberg. Ladyman seems to catch the correct solution:

> The idea then is that we have various representations which may be transformed or translated into one another, and then we have an invariant state under such transformations which represents the objective state of affairs.[34]

The idea goes back to the famous 'Erlangen Program' in geometry, formulated by Felix Klein in 1872. He classified all geometries existing at the time according to various symmetry groups, and defined a geometry as a theory of invariants with respect to a given group. This approach was taken up by Weyl in his analysis of relativity theory and quantum mechanics.

> For Weyl, appearances are open only for intuition (in the Kantian sense of subjective perception) and therefore agreement is obtained by giving objective status only to those relations that are invariant under particular transformations.[35]

The situation could be summarized in the following way. There is an abstract structure[36] which admits various representations: A, B, C, say. They are well defined mathematical structures. Let us suppose that these mathematical structures have been used to formulate the same physical theory, and all these formulations lead to correct empirical predictions. In such a case, all of them must have 'something in common', and consequently there must exist some 'translation rules' from A to B, from B to C, etc. If we change from one such representations to another, something must be preserved. Otherwise these mathematical structures could not have been representations of the same theory. It is not important whether we are able to identify these 'representation invariants', or not. The vital

[34] J. Ladyman, *art. cit.*, p. 421.
[35] *Ibid.*, p. 420.
[36] Which can be interpreted in a Platonic or anti-Platonic way; this is not the subject-matter of the present paper.

point is that they do exist. And my claim is (like that of Ladyman) that precisely the collection of these 'representation invariants' is what a given theory is about, what constitutes the *ontology* of this physical theory. The situation is not unlike when the meaning of a book is defined as an invariant with respect to the family of all its possible translations: indeed, the meaning of a book is what does not change when we change from one its (good) translation to another its (good) translation.

If we believe in the success of physics, we are entitled to claim that the structures of the successive physical theories (in the above sense) approximate, or at least are somehow related, to the Structure of the World.